Chapter 9
Non-linear problems

9.1. Introduction

The investigated so far linearly-elastic analysis is valid when all the stresses in the structure are under the limit of proportionality, the displacements are small, the supports in the structure are stable and the loads keep their directions during the deformation of the structure. Under these conditions the system of equations $[K]\{\Delta\}=\{R\}$ may be written for the initial geometric and static conditions, and the displacements $\{\Delta\}=[K]^{-1}\{R\}$ are derived in one step of solving the equation.

In case,that the structure is being deformed elastically, but is subjected to significant displacements and its initial geometry varies significantly,the problem is considered to be geometrically non-linear. Because of the great displacements the equilibrium equations must be written for the deformed configuration of the structure. A classical example for this is the transverse drop of a slab, where it turns to a slightly bended shell and opposes the transverse loading in membrane way like an inflated balloon. Another example like this is shown on fig. 9.1, a, where the force “follows” the deformed shape of a thin beam,and the relationship $F-v$ turns to non-linear. There are cases, where during the loading, big deformations could occur and then we should use the relationships between strains-displacements according to equations (1,14),

When the stresses in part from the material of the structure become larger than the point of the yield strength .the problem is non-linear with respect to the material. (fig. 9.1, b). A geometrically and materially combined problem is obtained, when the strains or the displacements are great and at the same time the material possess non-linear properties.

A non-linear problem is derived also in changing the contact spot with variation of the load. (fig. 9.1, c), because the relationship between the force and the contact stresses becomes non-linear

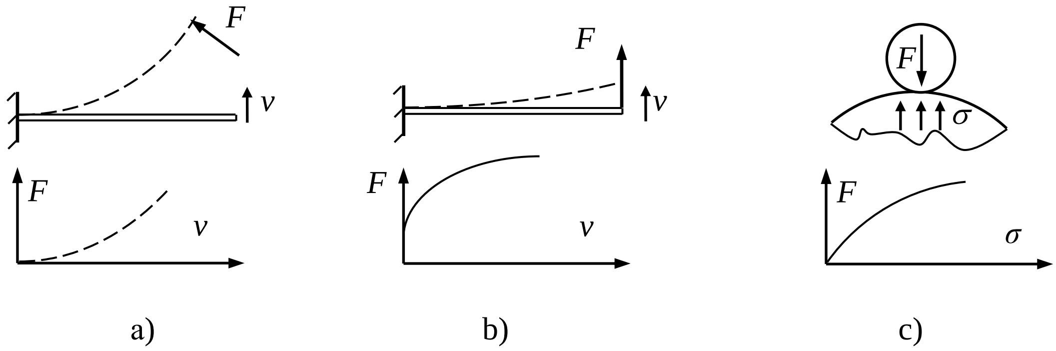
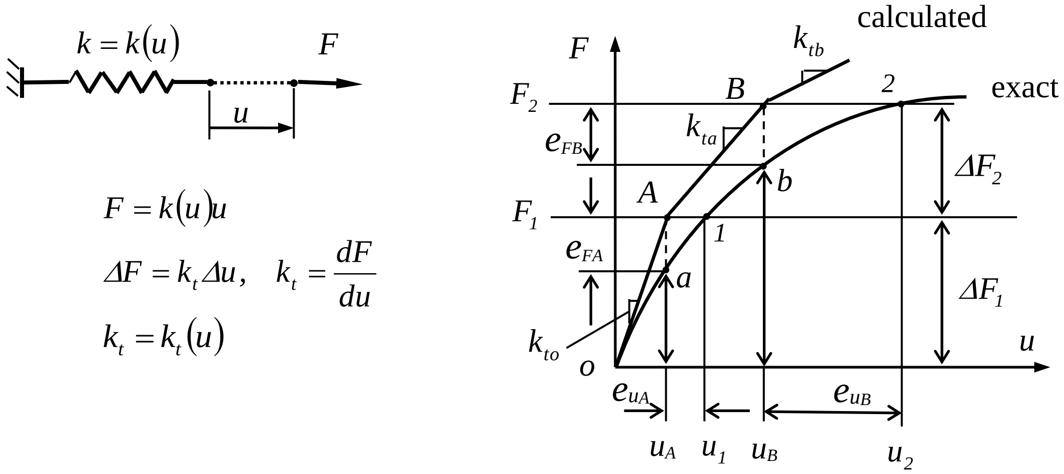


fig. 9.1

The solution of the non-linear problems can be obtained in one step, since the equations consist of conditions, not known till finding the solution. These can be ,the real configuration of the structure in a given moment , the loading forces, the stress state, the geometric boundary conditions. The solution is obtained in steps, like on each step the trial is revised and is repeated any number of times until the test for similarity is satisfied.

9.2. Algorhythm for numerical solution

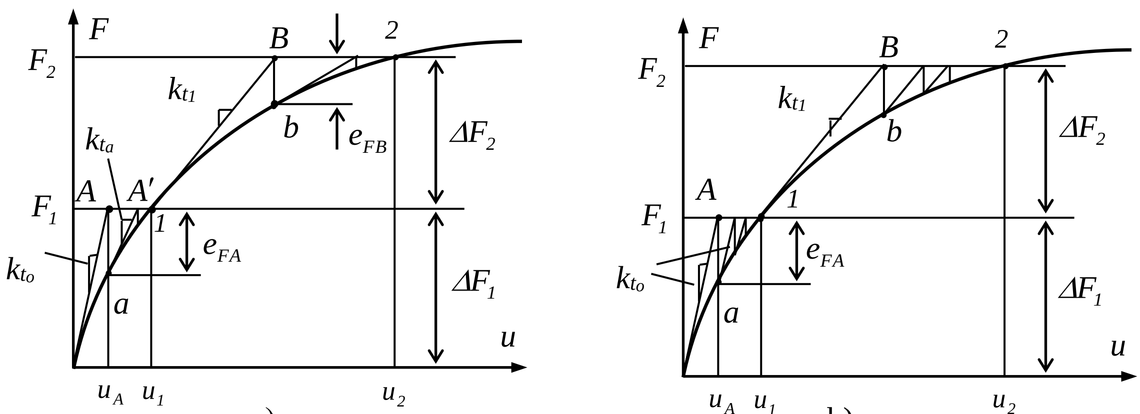
Many from the features of the methods used in solving the non-linear problems appear to be one and the same, nevertheless the source of non-linearity. The easiest way to explain the algorhythm for solving the problems is in investigating a system with one degree of freedom (fig. 9.2, a). The system consists of one non-linear spring. If we want to determine the displacement u for every value of the force F ,then obviously it can not be done directly, since $k=k(u)$.



a) b)

fig. 9.2

The exact solution (fig. 9.2, b) can be derived from calculation of F for each from the consecutive values of the displacement u. The approximated solution is obtained in steps, using the so called tangent stiffness $k_t = {dF}/{du}$. We begin with the initial tangent stiffness k_{to} when $u=0$. With the increase of the force ΔF_1 it is reached point A, for which the displacement is $u_A = u_a$. The tangent stiffness corresponding to that point is k_{ta}. TH linear equation for this step is $k_{to}\Delta u_1 = F_1 - 0$. It can be solved with respect to the displacement Δu_1 and then $u_A = 0 + \Delta u_1$. On the next step, the increase of the force is ΔF_2, and the tangent stiffness is k_{ta}. The linear equation $k_{ta}\Delta u_2 = F_2 - F_1$ is solved with respect to Δu_2 so $u_B = u_A + \Delta u_2$. So, the numerical solution can be obtained from connecting the linear segments of the separate steps. Practically, the exact solution is not known on fig. 9.2, b too, it is shown so it can be seen that the calculated curve has progressive deviation from the exact one. The calculated displacements u_A, u_B and so on have an error $e_к$, e_{uB} and so on. The methods for correction of these errors are illustrated in figures 9.3, a and 9.3, b.



a) b)

fig. 9.3

From fig. 9.3, a is seen that on the first step of the solution the force misbalance is e_{FA}. This misbalance, according to the Newton-Raphson method is used for correcting the displacement to the correct value u_1 done with iterations using the new tangent stiffness k_{ta} for point A, then for A' and so on. With k_{ta} using the equation $k_{ta}\Delta u = e_{FA}$ we obtain Δu, that is added to u_A so we get to point A'. In this point the force misbalance is already smaller. With the new tangent stiffness for point A' is made a correction of the displacement. Each of the following iterations decreases the force disbalance. The iterations continue until it reaches a certain predefined value or below it, that satisfies the given requirements for similarity. ON the next step a new increase of the force F_2 is given reaching point B, where new iterations are done. So using some points from the curve $F - u$ we get an approximation of the real curve.

A disadvantage of this method for systems with many degrees of freedom is that in the iterations one must continuously calculates new tangent stiffness matrix $[K_t]$, that may costs a lot of machining time. In the modified Newton-Raphson method (fig. 9.3, b) the iterations in each level of the force are done with one and the same stiffness matrix,defined in the beginning.In that case,though, the number of the necessary iterations is significantly large.

There another methods for numerical solution, but no matter the method, the final result is reduced to configuration $[\Delta]$, where the applied forces are in balance with the resistance of the structure.

The iterations on a given level of the forces stop at the moment when the solution is close enough to the initially accepted error. A possible definition of an error in the case of a problem for a system with many degrees of freedom is the equation

$$E = \frac{\|E_F\|}{\|R\|}, \tag{9.1}$$

where: $\|E_F\|$ and $\|R\|$ usually are Euclid's forms and $\|E_F\| = \sum \left(\Delta R_i^2\right)^{1/2}$, ΔR_i is a current force misbalance of i^{th} node, and $\|R\| = \sum \left(R_i^2\right)^{1/2}$, R_i is a currents load applied to i^{th} node. The iteration of i^{th} level can be stopper if the error has reached for example 0,001. The choice of this number is an important question since the very small tolerance can lead to usage of time for unnecessary accuracy or vice versa ,the big tolerance can not give the necessary accuracy of the results.

9.3. Stiffing in bending

The effect comes from the membrane stresses in the structure subjected to bending with great transverse displacements. The membrane stresses from compression decrease the stiffness of bending in trusses, slabs and shells, and the tension – increase it. In very big membrane stresses the stiffness of bending can become zero and the structure to loose stability. In such cases, in the structure occurs a transformation of the strain energy of the deformations from the membrane stresses in equal in value strain energy of the stresses in bending, without the applied loads to be changed. The critical state of the structure comes in the moment when it is impossible an easy variation of the strain state with "flowing" of strain energy from the membrane stresses in energy of stresses during bending. Therefore in a thin beam the axial stiffness AE/L is very larger than the stiffness in bending EJ/L^3, so in the thin walled structures the membrane stiffness is usually of times greater than the stiffness of bending. That means, that small membrane stresses can support big strain energy and in its flowing in stress energy from bending to obtain large deformations in the structure.. The stiffing usually is of a great importance for the thin walled structures, while in the massive ones in is negligibly small.

That effect of stiffing from the membrane stresses is defined by the matrix $[k_г]$, called geometric stiffness matrix. Another names are: stiffing matrix of membrane stresses (therefore in can be expressed as $[k_\sigma]$), initial stiffness matrix, matrix of the stability coefficient. As you can see later, this matrix do not depends on the material properties ,but it can change the conventional stiffness matrix. The obtaining of the global stiffness matrix $[K_г]$ of the structure is done in the same way as for the global one $[K]$.

9.3.1. Geometric stiffness matrix for truss element

In order to define the geometric stiffness matrix for a truss element, we need to determine the transverse displacement(fig. 9.4).

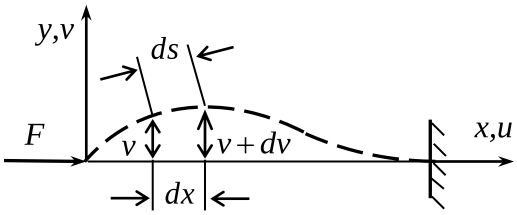


fig. 9.4

According to fig. 9.4 following the Pitagor's theorem, we can write

$$(ds)^2 = (dv)^2 + (dx)^2 \text{ или } \frac{ds}{dx} = \left[1 + \left(\frac{dv}{dx}\right)\right]^{1/2}. \tag{9.2}$$

If (9.2) is derived in a row, we get

$$\frac{ds}{dx} = 1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{1}{8}\left(\frac{dv}{dx}\right)^4 + \dots , \tag{9.3}$$

and after neglecting the additives from greater order

$$\frac{ds}{dx} = 1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 . \tag{9.4}$$

from *ds* we obtain adding relative deformation

$$\varepsilon_x^{/} = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \tag{9.5}$$

then the full relative deformation is

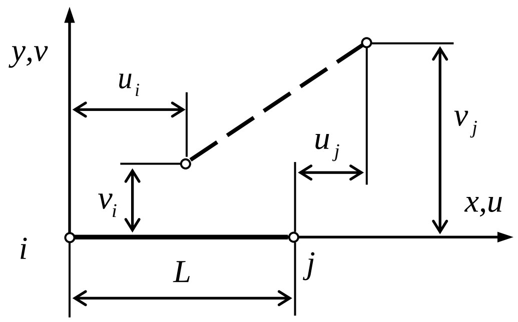
$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 . \tag{9.6}$$

If for approximating the displacements in the both directions are used one and the same functions of the shape we can write(fig. 9.5)

$$\begin{aligned} u &= \left(1 - \frac{x}{L}\right)u_i + \frac{x}{L}u_j \\ v &= \left(1 - \frac{x}{L}\right)v_i + \frac{x}{L}v_j \end{aligned} . \tag{9.7}$$

The strain energy for the element is derived from the equation

$$U_e = \frac{AE}{2}\int_0^L \left[\left(\frac{du}{dx}\right)^2 + \frac{du}{dx}\left(\frac{dv}{dx}\right)^2 + \frac{1}{4}\left(\frac{dv}{dx}\right)^4\right]dx . \tag{9.8}$$



Фиг. 9.5

Substitute (9.7) in (9.8) and after neglecting the values from greater order we get

$$U_e = \frac{AE}{2}\int_0^L \left[\left(\frac{-u_i + u_j}{L}\right)^2 + \left(\frac{-u_i + u_j}{L}\right)\left(\frac{-v_i + v_j}{L}\right)^2\right] dx\,. \tag{9.9}$$

After doing the calculations ,finally we can write

$$U_e = \frac{AE}{2L}\left(u_i^2 - 2u_i u_j + u_j^2\right) + \frac{F}{2L}\left(v_i^2 - 2v_i v_j + v_j^2\right), \tag{9.10}$$

where $F = \frac{AE}{L}\left(u_i - u_j\right)$.

The conditions for minimization of the full strain energy give us

$$\begin{array}{ll} \frac{\partial U_e}{\partial u_i} = \frac{AE}{L}\left(u_i - u_j\right) & \frac{\partial U_e}{\partial u_j} = \frac{AE}{L}\left(-u_i + u_j\right) \\ \frac{\partial U_e}{\partial v_i} = \frac{F}{L}\left(v_i - v_j\right) & \frac{\partial U_e}{\partial u_i} = \frac{F}{L}\left(-v_i + v_j\right) \end{array}, \tag{9.11}$$

The equations (9.11) may be written in matrix form as follows

$$\left(\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{F}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \left(\left[\bar{k}_k\right] + \left[\bar{k}_г\right]\right)\{d\} = \left[\bar{k}\right]\{d\}, \tag{9.12}$$

$$\text{where } \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\bar{k}_k\right] \text{ and } \frac{F}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \left[\bar{k}_г\right] \tag{9.13}$$

are the stiffness matrix of the truss element and the so called geometric stiffness matrix in the local coordinate system. For 2D truss element (fig. 3.3, b) the transforming of the local geometric stiffness matrix in a global one is done with the transforming matrix (3.22),as $\left[k_г\right] = \left[T\right]^T\left[\bar{k}_г\right]\left[T\right]$. The picture is analogical in 3D truss element.

9.3.2. Geometric stiffness matrix for a beam element

It is used for analysis of axial loaded columns and stiff connected planar and space truss structures. The loading could be as with axial so as with transverse forces and couples.

For a such element (fig. 9.6), when using (3.11) and the shape functions in (3.68), the interpolation functions are

$$u = u_1(1-\xi) + u_2\xi$$
$$v = v_1\left(1-3\xi^2+2\xi^3\right) + \theta_1 L\left(\xi-2\xi^2+\xi^3\right) + v_2\left(3\xi^2-2\xi^3\right) + \theta_2 L\left(-\xi^2+\xi^3\right), \quad (9.14)$$

where $\xi = x/L$.

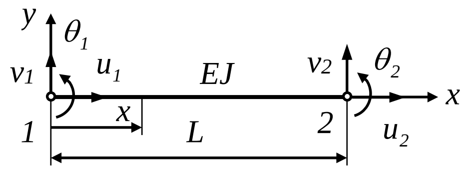


fig. 9.6

When we consider the additional axial deformation from the significant drop, for the relative deformation on a distance y from the axis we could write

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{d^2v}{dx^2}y. \quad (9.15)$$

For the strain energy could be written

$$U = \int_V \frac{1}{2}E\varepsilon_x^2 dV = \int_0^L \int_A \frac{1}{2}E\varepsilon_x^2 dxAd. \quad (9.16)$$

After substitution of (9.15) in (9.16) and taking into account

$$\int_A dA = A, \quad \int_A zdA = 0, \quad \int_A z^2 dA = J, \quad \int_A E\frac{\partial u}{\partial x}dA = F, \quad (9.17)$$

We obtain

$$U = \int_0^L \frac{AE}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx + \int_0^L \frac{F}{2}\left(\frac{\partial v}{\partial x}\right)^2 dx + \int_0^L \frac{EJ}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx \quad (9.18)$$

In (9.18) F is the axial force, positive in tension. The quantity $\left(\frac{\partial v}{\partial x}\right)^4$ is neglected as too small from a greater order.

The first integral in (9.18) leads to $[k]$ for truss element related to degrees of freedom u_1 and u_2. The third integral leads to $[k]$ for a beam element related to degrees of freedom v_1, θ_1, v_2 and θ_2. The second integral gives us the geometric stiffness matrix $[k_z]$. It corresponds to the work done and the accumulated strain energy for transverse displacement v in the presence of constant axial force.

In nodal elements $\{d\}^T = [u_1\ v_1\ \ u_2\ v_2]$ the transverse displacement and its derivative is

$$[v] = [N]\{d\}, \text{ where } [N]^T = [N_1\ N_2\ N_3\ N_4] \quad (9.19)$$

and, where $$[G]^T = \left[\frac{\partial N_1}{\partial x}\ \frac{\partial N_2}{\partial x}\ \frac{\partial N_3}{\partial x}\ \frac{\partial N_4}{\partial x}\right]. \quad (9.20)$$

Now the second integral from (9.18) is defined as

$$\int_0^L \frac{F}{2}\left(\frac{\partial v}{\partial x}\right)^2 dx = \frac{1}{2}\int_0^L \left[\frac{\partial v}{\partial x}\right]^T F\left[\frac{\partial v}{\partial x}\right]dx = \frac{1}{2}\{d\}^T[k_z]\{d\}, \quad (9.21)$$

where $$[k_г] = \int_0^L [G]^T F [G] dx . \qquad (9.22)$$

In (9.22) the force is constant and can be put out from the integral. Using standard functions of the shape (3.68) for the geometric stiffness matrix we receive

$$[\bar{k}_г] = \frac{F}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & 3L & -L^2 \\ & & 36 & 3L \\ сим. & & & 4L^2 \end{bmatrix} . \qquad (9.23)$$

After expanding of the geometric matrices (9.13) and (9.23) with zero columns and rows to 6x6, we can obtain the matrix for an element with nodal parameters $\{d\}^T = [u_1\ v_1\ \theta_1\ u_2\ v_2\ \theta_2]$

$$[\bar{k}_г] = \frac{F}{30L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 36 & 3L & 0 & -36 & 3L \\ & & 4L^2 & 0 & -3L & -L^2 \\ & & & 0 & 0 & 0 \\ & & & & 36 & -3L \\ \text{symmetrical} & & & & & 4L^2 \end{bmatrix} , \qquad (9.16)$$

9.3.3. Geometric stiffness matrix for a slab element

It is derived by defining the work of constant membrane forces (fig. 9.7) for transverse displacements related to the drop.

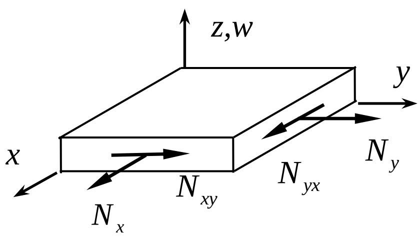


fig. 9.7

The membrane forces are defined according to (7.2), and the membrane deformations,related to small rotations $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are defined according to (1.17). If we assume that the membrane forces N_x, N_y and N_{xy} are independent on small transverse displacements $w = w(x,y)$, the strain energy of the deformations will be

$$U = \int_A \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 N_x + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 N_y + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} N_{xy} \right] dA , \qquad (9.17)$$

Or in matrix form $$U = \frac{1}{2}\iint \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix}^T \begin{bmatrix} N_x & N_{yx} \\ N_{xy} & N_y \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} dx dy = \frac{1}{2}\{d\}^T [k_г] \{d\} . \qquad (9.18)$$

The rotations can be defined by the displacement field $w = [N]\{d\}$ or

$$\begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} = [G]\{d\}, \tag{9.19}$$

as $[G]$ is with dimensions *2 x n,* and $\{d\}$ *n x 1,* as *n* is number of the degrees of freedom of the element. Now the geometric stiffness matrix is:

$$[k_г] = \iint [G]^T \begin{bmatrix} N_x & N_{yx} \\ N_{xy} & N_y \end{bmatrix} [G] dx dy. \tag{9.20}$$

The matrix $[G]$ in the common case is a function of *x* and *y,* and the integrations is done on for the surface of the element.

9.4. Linear analysis in loss of stability

It is based on the matrix equation for a system with many degrees of freedom

$$|[K] + \lambda[K_г]|\{\delta\Delta\} = 0, \tag{9.20}$$

where $[K_г]$ is calculated for arbitrary level of the membrane stresses, λ is a factor, with the help of which that level can be increased or decreased in order to loose the stability, $\{\delta\Delta\}$ is infinitely small variation of the displacement vector. We are looking for a solution different than $\{\delta\Delta\} = 0$. The case corresponds on the so called classical analysis of loosing the stability. In the critical state, two different forms are possible to occur, without variation of the applied loads $\{R\}$. The increase of the displacements $\{\delta\Delta\}$ begins from the configuration $\{\Delta\}$, exactly before the loss of the stability. From the equations (9.20) obviously, we get to the problem for the natural quantities. The calculated value of λ can be with positive or negative sign, depending on the membrane stresses, used for the creation of the matrix $[K_г]$.

A good example for this is the classical one with the column shown on fig. 9.8.

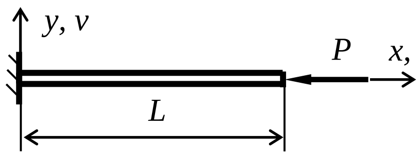


fig. 9.8

The column may be modeled by one element and then $[K]$ and $[K_г]$ are defined according (3.82) and (9.16). If we accept that $F = -1$ and taking into account that $u_1 = v_1 = \theta_1 = 0$ using (9.20) we get

$$\left(c_2 \begin{bmatrix} c_1/c_2 & 0 & 0 \\ 0 & 12 & -6L \\ 0 & -6L & 4L^2 \end{bmatrix} + \lambda_{кр} \frac{-1}{30L} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3L \\ 0 & -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{9.21}$$

where $c_1 = AE/L$ and $c_2 = EJ/L^3$. In order to obtain a solution different than $u_2 = v_2 = \theta_2 = 0$, the determinant of the expression in the brackets must be set to zero.The lowest value of $\lambda_{кр}$ is $\lambda_{кр} = 2{,}486\, EJ/L^2$, where for the critical force we obtain

$$F_{кр} = \lambda_{кр}(-1) = -2{,}486\, EJ/L^2 . \tag{9.22}$$

The geometric stiffness matrix in (9.21) may be taken from (9.13) and then for $F_{кр}$ we have $F_{кр} = -3EJ/L^2$. This value is not so exact as the one from the result in (9.22).In both cases the obtained value for $F_{кр}$ is greater than the exact one that is $F_{кр} = -2{,}4674 EJ/L^2$.

The bended shape can be defined from (9.21) with $\lambda = \lambda_{кр}$, choosing arbitrary value of one of the degrees of freedom (for example $\theta_2 = 1$) and then calculate the other 2 Thus we get

$$u_2 = 0, \quad w_2 = 0{,}6379L, \quad \theta_2 = 1 . \qquad (9.23)$$

The defined displacements are with respect to the non-deformed shape of the structure. The membrane stresses are not changed as distribution, but are only scaled by the factor of λ in order to foresee the reach of the critical state.

The linear analysis for loss of stability frequently gives results bigger than the real value of the critical load. It works good with straight columns and slabs, where we do not have bending till loss of stability. In the most thin walled structures the membranes and the bending stresses are developed at one and the same time and interact on each other before the initiation of the critical state. Thus the membrane stresses can be changed during the process of loading nonlinearly from the loads. Moreover the matrix $[K]$ becomes a function of the displacements, if they are large or there are plastic deformations. That’s why most of the practical cases of loss of stability are nonlinear problems and must be based on the tangent stiffness matrix.