Chapter 8
Dynamic Problems

8.1. Introduction

Like it has been already mentioned at the static action of the loads, the loading is characterized with its slow variation until it reaches its final value. Many constructions in the engineering practice are dealing with loads, which vary with time, during which it is possible the change to be a periodic one. If the frequency, with which the load is varying is smaller than one third of the lowest natural frequency of the construction, the effect of the inertia forces can be neglected and the problem is considered as a quazy-static, which means, that the analysis can be done like if we had a static loading. With higher frequencies the inertia forces become significant and they should be considered. The mechanical processes in the constructions usually are combined with internal friction, which complicates the analysis. One of the important aspects of this, so called dynamic analysis, is the determination of the natural frequencies and their forms. The goal is not to allow the natural frequency of the construction to be closer to that of the source of oscillation, which can cause a resonance. Another very important aspect of the dynamic analysis is the determination of the stresses, deformations and displacements in the construction with time and the goal is to make a register of the values at curtain points for a given period of time.

When we have extreme loading in the construction, elastic waves with high frequencies occur, which propagate for a very short period of time. The problem in this case is to analyze the propagation of the waves in the construction.

8.2. Fundamental dependencies. Damping and Mass matrixes.

8.2.1. Forced oscillation of a mass with one degree of freedom

Shown on fig.1, a one-mass system, which is subjected to a forced oscillation due to force $F = F(t)$.

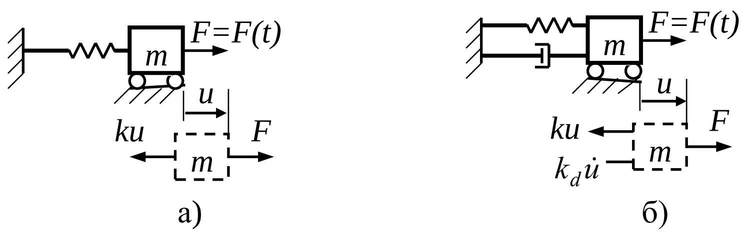


Fig. 8.1

According to Dalamber's principle it can be written

$$F = ku + m\ddot{u}\,, \tag{8.1}$$

where κ is a spring constant, $u = u(t)$ is the only degree of freedom, $\ddot{u} = \dfrac{d^2u}{dt^2}$.

When we have damping (fig. 8.1, б) the equation is

$$F = ku + k_d\dot{u} + m\ddot{u}\,, \tag{8.2}$$

where k_d in the general case is a damping parameter, $\dot{u} = \dfrac{du}{dt}$.

8.2.2. Free oscillation of a system with one degree of freedom

We have free oscillation of the system when $F(t)=0$. Without damping the equation is

$$ku + m\ddot{u} = 0\,. \tag{8.3}$$

The solution of (8.3) is

$$u = \bar{u}\, sin\, \omega t\,, \tag{8.4}$$

where $\bar{u}$ is the amplitude of the displacement (fig. 8.2, a), ω is the natural frequency(*rad/sec*). The frequency (*Hz*) is calculated from

$$f = \frac{\omega}{2\pi}\,. \tag{8.5}$$

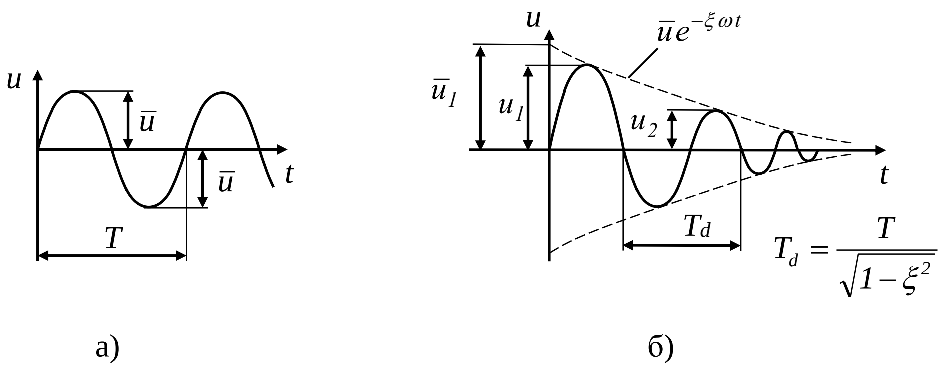


Fig. 8.2

On fig. 2, *T* is oscillation period.

After replacement of (8.4) in (8.3) we get

$$\omega = \sqrt{\frac{k}{m}}\,. \tag{8.6}$$

When we have damping (fig. 2, б) and $F=0$ we get oscillation with circular frequency

$$\omega_d = \omega\sqrt{1-\xi^2}\,, \tag{8.7}$$

where $\xi = \frac{k_d}{k_c}$ is the so called damping ratio and $k_d < k_c$. Critical value of the damping coefficient k_c for oscillating motion is

$$k_c = 2m\omega\,. \tag{8.8}$$

With real constructions the damping is small, $\xi < 0{,}15$ and $\omega_d \approx \omega$.

8.3. Oscillation of a system with many degrees of freedom

The energy balance of an element from a discrete system can be written with the equation

$$\begin{aligned}&\int_V \{\delta u\}^T \{R_V\} dV + \int_A \{\delta u\}^T \{p\} dA + \sum_{i=1}^{n} \{\delta u\}_i^T \{P\}_i = \\ &= \int_V \left(\{\delta\varepsilon\}^T \{\sigma\} + \{\delta u\}^T \rho \{\ddot{u}\} + \{\delta u\}^T k_d \{\dot{u}\} \right) dV \quad ,\end{aligned} \tag{8.9}$$

having in mind, that with small virtual displacements, the work done by the volumes $\{R_V\}$, the surfaces $\{p\}$ and the concentrated forces $\{P\}_i$ is equal to the work done by the internal, inertia and damping forces. In (8.9) ρ is the mass density.

The fields of the displacements, the velocities and the accelerations can be approximated according to the dependency

$$\{u\} = [N]\{d\}, \quad \{\dot{u}\} = [N]\{\dot{d}\}, \quad \{\ddot{u}\} = [N]\{\ddot{d}\}, \tag{8.10}$$

where the functions of the form $[N]$ are only space functions, while the nodal degrees of freedom $\{d\}$ are only time functions. From (8.10) and (8.9) we get

$$\{\delta d\}^T [\int_V [B]^T \{\sigma\} dV + \int_V \rho [N]^T [N] dV \{\ddot{d}\} + \int_V k_d [N]^T [N] dV \{\dot{d}\} -$$

$$- \int_V [N]^T \{R_V\} dV - \int_A [N]^T \{p\} dA - \sum_{i=1}^{n} \{P\}_i]=0. \quad (8.11)$$

in (8.11) A the area, on which the surface forces act, while the forces $\{P\}_i$ can be considered as concentrated at the nodes. The equation (8.11) can be written also in this form

$$[c]\{\dot{d}\} + [m]\{\ddot{d}\} + \{r_{вт}\} = \{r_{вн}\}, \quad (8.12)$$

where

$$[r_{вт}] = \int_V [B]^T \{\sigma\} dv \quad (8.13)$$

is a vector of the generalized internal forces,

$$[m] = \int_V \rho [N]^T [N] dV \quad (8.14)$$

is the so called mass matrix, while

$$[c] = \int_V k_d [N]^T [N] dV \quad (8.15)$$

is the damping matrix.

The vector of the generalized external nodal forces, is defined by the expression

$$\{r_{вн}\} = \int_V [N]^T \{R_V\} dV + \int_A [N]^T \{p\} dA + \sum_{i=1}^{n} \{P\}_i \ . \quad (8.16)$$

This system of equations (8.12), they are ordinary differential equations from second order, in which the displacements $\{d\}$ are discrete functions in space and at the same time continuous functions in time. In dynamic analysis there are two ways for solving them. In the so called module methods the solution is done by separating the equations on space and time parameters and after that their independent consideration. The other methods are methods of direct integration, in which the equations (8.12) are made discrete in time, so we can get a sequence of separate systems of algebraic equations.

The summarized mass and damping matrixes $[M]$ and $[C]$ for the whole system obtained by analogical manner as we obtained the stiffness matrix, when the matrixes for the different elements in the global coordinate system are expanded with the size of the summarized matrix and are summed after that.

The vector of the internal forces (8.13) gives the nodal loads, caused by the material deformation. The dependencies (8.12) and (8.13) are valid for linear and non-linear behavior of the material.

With linearly-elastic behavior of the material $\{\sigma\} = [E][B]\{d\}$ and the equation (8.13) becomes

$$\{r_{вт}\} = [k]\{d\}. \quad (8.17)$$

Now (8.12) it can be written in the form

$$[k]\{d\}+[c]\{\dot{d}\}+[m]\{\ddot{d}\}=\{r_{вн}\}. \tag{8.18}$$

For the assembled system according to (8.18) it can be written

$$[K]\{\Delta\}+[C]\{\dot{\Delta}\}+[M]\{\ddot{\Delta}\}=\{R_{вн}\}, \tag{8.19}$$

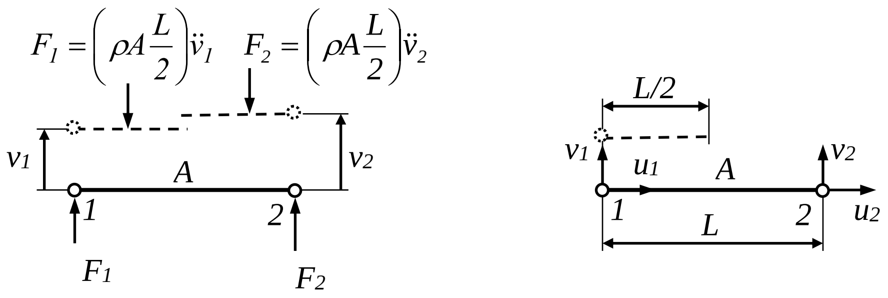
where $\{R_{вн}\}$ is a function of time.

If when we determine $[m]$ and $[c]$ the same functions of the form $[N]$ are used, with which the displacement field is approximated, the obtained matrixes are called consistent. With ρ and κ_d different from zero, the matrixes $[m]$ and $[c]$ are positive and symmetric.

8.4. Determination of the mass matrix

8.4.1. Diagonal matrix of the masses

We get the matrix directly by dividing the mass to parts and putting them to the nodes of the element. For truss element with one degree of freedom in node $\{d\}^T=[v_1\ v_2]$(fig. 8.3, a) the mass of the truss is represented in two parts $\rho A\dfrac{L}{2}$.



a) б)

Fig. 8.3

According to Newton's Law

$$F_1=\rho A\frac{L}{2}\ddot{v}_1 \text{ и } F_2=\rho A\frac{L}{2}\ddot{v}_2. \tag{8.20}$$

In matrix form (8.20) it can be written

$$\begin{bmatrix}\rho A\dfrac{L}{2} & 0\\ 0 & \rho A\dfrac{L}{2}\end{bmatrix}\begin{Bmatrix}\ddot{v}_1\\ \ddot{v}_2\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\end{Bmatrix} \text{ or } [m]\begin{Bmatrix}\ddot{v}_1\\ \ddot{v}_2\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}. \tag{8.21}$$

This matrix of masses $[m]$ has diagonal form.

The general plane motion with vector of the nodal displacements $\{d\}^T=[u_1\ v_1\ u_2\ v_2]$ (fig. 8.3, б) the matrix of the masses also has diagonal form

$$[m]=\rho A\frac{L}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}. \tag{8.22}$$

If for two-node truss element $v=v(x)$ (fig.8. 4) the elementary inertia force is

$$dF=\rho A\ddot{v}(x)dx=q(x)dx. \tag{8.23}$$

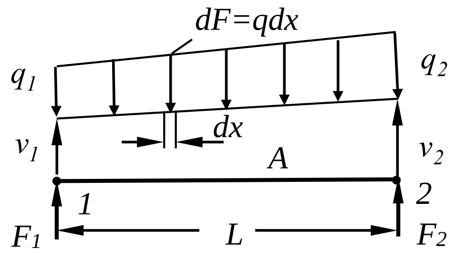


Fig. 8.4

With the reduction of the distributed load to the nodes we get

$$F_1 = \rho AL\left(\frac{1}{3}\ddot{v}_1 + \frac{1}{6}\ddot{v}_2\right) \text{ and } F_2 = \rho AL\left(\frac{1}{6}\ddot{v}_1 + \frac{1}{3}\ddot{v}_2\right). \tag{8.24}$$

In matrix form it is

$$\begin{bmatrix} \rho A L/3 & \rho A L/6 \\ \rho A L/6 & \rho A L/3 \end{bmatrix}\begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \text{ or } [m]\begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}. \tag{8.25}$$

For 2D beam element (fig. 8.5) with vector of the nodal displacements $\{d\}^T = [v_1\ \theta_1\ v_2\ \theta_2]$ the mass matrix represented by parts is also diagonal.

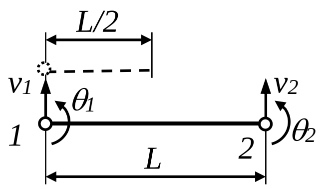


Fig. 8.5

If we take in mind, that the mass inertia moment of a part with length *L/2* is $J = \frac{\alpha L^2}{210}$,

The mass matrix can be presented in this form

$$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\alpha L^2}{210} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\alpha L^2}{210} \end{bmatrix}, \tag{8.26}$$

where α is a coefficient, which depends on the form. When $A = const$ $\alpha = 17{,}5$. The physical sense of the element $\frac{\alpha L^2}{210}$ of the matrix $[m]$ is a moment, which creates a unit angular acceleration of the part with length *L/2* on axis at one end of the part. The mass matrix for beam element with $\{d\}^T = [u_1\ v_1\ \theta_1\ u_2\ v_2\ \theta_2]$ can be obtained by expansion of the mass matrixes (8.21) and (8.25) and after that combining them.

The matrixes (8.21) and (8.25) are obtained easily thanks to their clear physical sense and are widely used.

8.4.2. "Consistent" mass matrix

It is obtained according to (8.14), by using the functions of the form for approximation of the displacements in the element. This matrix is not diagonal. As an example for this matrix we can consider the truss element from fig. 8.3, б with $\{d\}^T = [u_1\ v_1\ u_2\ v_2]$. For approximation of the displacements in the two directions the same functions of the form are used $N_1 = 1 - \frac{x}{L}$, $N_2 = \frac{x}{L}$. After substitution in (8.14) we get

$$[m] = \int_0^L \rho \begin{bmatrix} 1-\frac{x}{L} & 0 \\ 0 & 1-\frac{x}{L} \\ \frac{x}{L} & 0 \\ 0 & \frac{x}{L} \end{bmatrix} \begin{bmatrix} 1-\frac{x}{L} & 0 & \frac{x}{L} & 0 \\ 0 & 1-\frac{x}{L} & 0 & \frac{x}{L} \end{bmatrix} A dx = \frac{m}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}, \qquad (8.27)$$

where $m=\rho AL$.

For the beam element from fig. 8.5 with $\{d\}^T = [v_1\ \theta_1\ v_2\ \theta_2]$ and approximation functions according to (8.68) times (8.14) for the mass matrix we get

$$[\mathrm{m}] = \frac{\rho \mathrm{AL}}{420} \begin{bmatrix} 156 & 22\mathrm{L} & 54 & -13\mathrm{L} \\ & 4\mathrm{L}^2 & 13\mathrm{L} & -3\mathrm{L}^2 \\ & & 156 & -22\mathrm{L} \\ \text{symmetr.} & & & 4\mathrm{L}^2 \end{bmatrix}. \qquad (8.28)$$

In analogical manner we can make the mass matrixes for 2D and 3D final elements. There are schemes, with which the "consistent" matrix can be transformed into diagonal, at the same time the characteristics of the full matrix can be retained.

All the mass matrixes considered so far, represent the resistance of the element to translational acceleration, never mind their formulation. The difference is in the way of the formulation of the resistance to angular acceleration, this explains the different characteristics and advantages of the matrixes. The different strategies in the different cases determine which of them or a combination of both is going to be used to solve the problem. It doesn't matter if the mass matrix is diagonal or full (if we have or we don't have inertia moment connected with the degree of freedom and angular acceleration), the resemblance of the solution with the improved mesh is kept, if the mass matrix gives exact values for the inertia forces as an answer to the virtual translation acceleration of the element. If the elements are compatible and the numerical integration is full with the "consistent" matrix, the calculated frequency of the FEA model is surely greater than that of the mathematical model. This means, that the resemblance of the solution is better when the mesh is finer.

8.5. Free oscillation without damping

When we do not have damping in equation (8.19) $[C]=0$, while $\{R_{вн}\}$ is either 0 or *const,* but only if they do not cause change in the stiffness matrix due to high stresses. Usually the acting loads are connected with masses and they should be considered in $[M]$.

With the oscillation analysis, the goal is the natural frequencies and forms to be determined. Let $\{R_{вн}\}=0$, then (8.19) becomes

$$[M]\{\ddot{\Delta}\}+[K]\{\Delta\}=0. \qquad (8.29)$$

The solution when we have harmonic motion of the construction without supports is

$$\{\Delta\} = \{\overline{\Delta}\} sin\, \omega t \text{ and } \{\ddot{\Delta}\} = -\omega^2 \{\overline{\Delta}\} sin\, \omega t\,, \tag{8.30}$$

where $\{\overline{\Delta}\}$ is the matrix of the amplitude values of the harmonic oscillation.
According to (8.30) the displacements in the nodes are sin-phase and have one and the same frequency.

After replacing (8.30) in (8.29) we get a system of equations

$$([K] - \lambda [M])\{\overline{\Delta}\} = 0\,, \tag{8.32}$$

where $\lambda = \omega^2$. The equation (8.32) is known as the general algebraic problem for the eigen values. If the matrix $[K] - \lambda[M]$ is nonsingular, there is only the trivial solution $\{\overline{\Delta}\} = 0$. We search for the interesting nontrivial solutions in

$$det([K] - \lambda [M]) = 0\,. \tag{8.33}$$

The values of λ, which satisfy (8.33) represent the squares of the natural frequencies of oscillation of the construction. For each eigen value λ_i there is an eigen vector $\{\overline{\Delta}\}_i$, which is called eigen form. The vector $\{\overline{\Delta}\}_i$ is also called matrix of the на i^{th} form of oscillation of the construction, because the displacements can be determined from the dependency

$$[u] = [N]\{d\}_i = [N]\{\overline{d}\}_i\, sin\, \omega t\,, \tag{8.31}$$

from here and the form of the deformed construction due to oscillation. The eigen values and vectors have important characteristics, orthogonal and linear independence.

Because the determinant of the system

$$([K] - \lambda_i [M])\{\overline{\Delta}\}_i = 0 \tag{8.32}$$

is zero, there is a linear dependence between the equations, which means, that one of the determined λ_i is a combination from the others. The system with n equations and n-1 unknowns is solved by assuming that one of the elements of $\{\overline{\Delta}\}_i$ is given, and the rest are determined from n-1 equations. This means, that the form oscillation can be determined very precisely, which can be seen from the fact, that when we multiply (8.32) with const multiplier nothing is changed.

Some of the more important aspects of the problem for determination of the natural frequencies and forms of the construction are:

- Here, in comparison with the static problem, we can determine the natural frequencies and forms of a construction without any supports. Because for this type of construction, displacements like an absolute solid body, are possible, during which the natural frequencies are zero and they must be rejected.
- Equation (8.32) can be written in the form $[K]\{\Delta\}_i = \omega_i^2 [M]\{\Delta\}_i$, which physically means, that the form of oscillation is a configuration, during which the loads and the inertia forces are in balance.
- The correlation between the elements of $\{\overline{\Delta}\}_i$ determines the i^{th} form and the amplitude is meaningless. Which means, that $\{\overline{\Delta}\}_i$ and $\{\overline{\Delta}\}_i$ are one and the same form, if c is any number except $c = -1$. Usually in the software products for FEA normalization is made for every form, during which the highest value of the displacement is one. We can get more general normalization from the dependency

$$\{\overline{\Delta}\}_i^T [M]\{\overline{\Delta}\}_i = 1\,. \tag{8.33}$$

It should be known, that the stresses determined using the normalized displacement, are with unreal high value.

- The eigen vectors are orthogonal in relation with the stiffness and mass matrixes, which means, that for every $i \neq j$ $\{\Delta\}_i^T [K]\{\Delta\}_j = 0$ and $\{\Delta\}_i^T [M]\{\Delta\}_j = 0$.
- Except when we have extreme loading, only the form of the lowest natural frequency is important to the considered construction.
- With the lower natural frequencies the length of the wave, which is distributed in the construction, is greater and the error from discretezation is smaller, because one wave can include more elements.

8.5.1. Example

Determine the lowest natural frequencies and forms of oscillation of the truss on fig 8.6. Given: *E, L, A.*

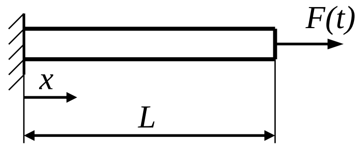


Fig. 8.6

The problem can be solved by assuming the truss as one element. Then according to (8.32), with matrix $[m]$ times (8.27) and baring in mind, that $u_1=0$, it can be written

$$\left(\frac{AE}{L}.1-\omega^2\frac{\rho AL}{6}.2\right)\bar{u}_2 = 0. \tag{8.34}$$

For nontrivial solution it is necessary the polynomial in the parenthesis to be zero, from where we get

$$\omega_1 = 1{,}732\left(\frac{E}{\rho L^2}\right)^{1/2}. \tag{8.35}$$

The exact solution with the mathematical model is $\omega_1 = 1{,}571\left(\frac{E}{\rho L^2}\right)^{1/2}$. The error is 10 %.

If we make a model with two elements according to (8.32) after assembling the stiffness and mass matrixes we get

$$\left(\frac{2AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2\frac{\rho AL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = 0. \tag{8.36}$$

If we assume $\frac{\omega^2 \rho L^2}{24E} = a$ the system becomes

$$\begin{bmatrix} (2-4a) & -(1+a) \\ -(1+a) & (1-2a) \end{bmatrix}\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{8.37}$$

For zero displacement amplitude the determinant should be zero

$$det\begin{vmatrix} (2-4a) & -(1+a) \\ -(1+a) & (1-2a) \end{vmatrix} = 0. \tag{8.38}$$

The solution of (8.38) gives: $a_1 = 0{,}108$ and $a_2 = 1{,}320$. From (8.37) we get both natural frequencies of oscillation

$$\omega_1 = (24a_1)^{1/2}\left(\frac{E}{\rho L^2}\right)^{1/2} = 1{,}610\left(\frac{E}{\rho L^2}\right)^{1/2}$$
$$\omega_2 = (24a_2)^{1/2}\left(\frac{E}{\rho L^2}\right)^{1/2} = 5{,}628\left(\frac{E}{\rho L^2}\right)^{1/2} \quad . \tag{8.39}$$

The error of the solution with this model is 2,5 %.

Because the system (8.37) using (8.38) is linearly dependant, it can be written

$$\bar{u}_2 = \frac{(1+a)}{(2-4a)}\bar{u}_3 . \tag{8.40}$$

So for the amplitude values of the displacements we get

$$a = 0{,}108 \quad \bar{u}_2 = 0{,}707\bar{u}_3$$
$$a = 1{,}320 \quad \bar{u}_2 = -0{,}707\bar{u}_3 \quad . \tag{8.41}$$

If we assume $\bar{u}_2 = 1$, for both forms of oscillation we get

$$\begin{Bmatrix}\bar{u}_1\\ \bar{u}_2\\ \bar{u}_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0{,}707\\ 1{,}0\end{Bmatrix}\bar{u}_3 \text{ and } \begin{Bmatrix}\bar{u}_1\\ \bar{u}_2\\ \bar{u}_3\end{Bmatrix} = \begin{Bmatrix}0\\ -0{,}707\\ 1{,}0\end{Bmatrix}\bar{u}_3 . \tag{9.42}$$

On fig 8.7 both forms of oscillation are shown, according to the exact solution and the solution with the FEA.

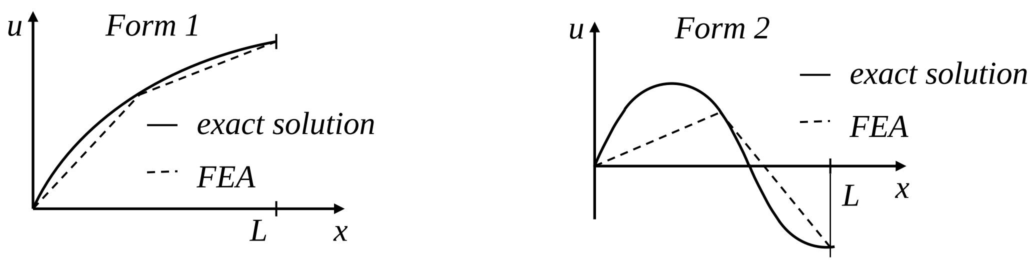


Fig. 8.7

8.6. Damping

The damping is a process of distribution of the energy in the construction, because of which the amplitude of the free oscillation drops with time. The process is complex and not well analyzed. Most frequently the damping is approximated with viscous model, when the damping forces are proportional to the velocity. In the real constructions the main sources of damping, like the internal friction in the material (hysteresis), have different characteristics and can't be represented in this way. But because the damping is with small value, the error from this representation is small too. For example the damping forces $[C]\{\dot{\Delta}\}$ are from the order of 10% of the forces $[K]\{\Delta\}$, $[M]\{\ddot{\Delta}\}$ and $\{R_{вн}\}$ in the equation (8.19). Idea for the damping gives the damping ratio ξ, which depending on the stresses in the construction is from 2% (when we have tube constructions with small diameter) up to 15% (when we have screws).

The most popular model of viscous damping is the Reley's model also called proportional damping. The matrix $[C]$ is represented as a linear function of the stiffness and mass matrixes

$$[C]=\alpha[K]+\beta[M], \tag{8.43}$$

where α and β are coefficients. So the damping matrix is orthogonal. The relationship between the damping ratio ξ and the frequency ω can be written in the form

$$\xi=\frac{1}{2}\left(\alpha\omega+\frac{\beta}{\omega}\right). \tag{8.44}$$

The dependency ξ - ω is shown on fig. 8.8. The damping, which corresponds to $\alpha[K]$ increases with the increase of the frequency, while this corresponding to $\beta[M]$ decreases with the increase of the frequency. It is discovered experimentally, that $\alpha[K]$ can represent the internal friction of the material, while $\beta[M]$ can represent the damping from the friction in the nodes.

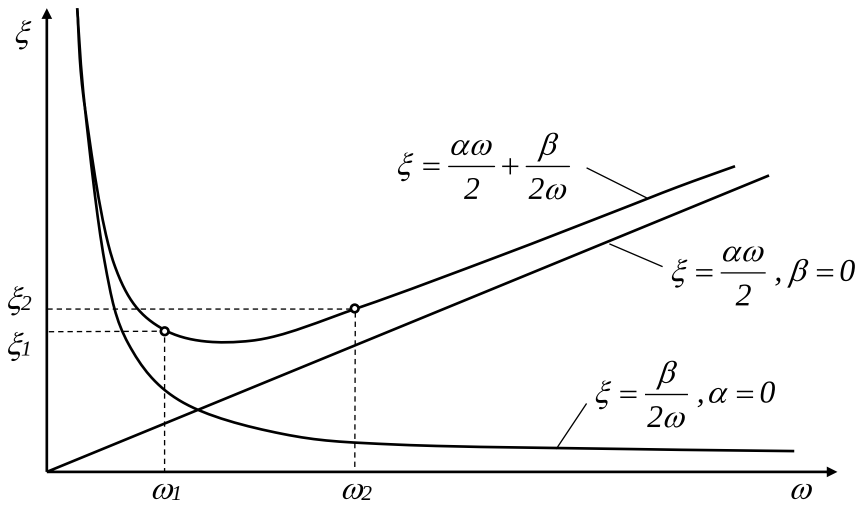


Fig. 8.8

With the chosen, from the designer of the construction, limits for the frequencies ω_1 and ω_2 (usually these frequencies are the highest and the lowest from those, that are of interest to the designer), coefficients α and β can be obtained from the equations

$$\begin{aligned}\alpha&=2(\xi_2\omega_2-\xi_1\omega_1)/(\omega_2^2-\omega_1^2)\\ \beta&=2\omega_1\omega_2(\xi_1\omega_2-\xi_2\omega_1)/(\omega_2^2-\omega_1^2).\end{aligned} \tag{8.45}$$

Except this between the damping ratio and the frequency for the different forms of oscillation it is valid that

$$2\xi_i\omega_i = \alpha + \beta\omega_i^2, \quad i = 1, 2 \ . \tag{8.46}$$

After determination of the coefficients α and β using (8.43) also a damping matrix is obtained.

8.7. Division of the motion by the forms of the natural frequencies. Module methods

If every eigen form is scaled according to (8.33) equation (8.32) it can be written

$$\{\overline{\Delta}\}_i^T [K]\{\overline{\Delta}\}_i - \omega_i^2 \{\overline{\Delta}\}_i^T [M]\{\overline{\Delta}\}_i = 0 \, . \tag{8.47}$$

Because $\{\overline{\Delta}\}_i^T [M]\{\overline{\Delta}\}_i = 1$ from (8.47) we obtain

$$\omega_i^2 = \{\overline{\Delta}\}_i^T [K]\{\overline{\Delta}\}_i \, . \tag{8.48}$$

Because the eigen vectors $\{\overline{\Delta}\}_i$ are orthogonal and $\{\overline{\Delta}\}_i^T [K]\{\overline{\Delta}\}_j = 0$, $\{\overline{\Delta}\}_i^T [M]\{\overline{\Delta}\}_j = 0$ with $i \neq j$, this can be written

$$[\Phi]^T [K][\Phi] = \left[\omega^2\right] \text{ и } [\Phi]^T [M][\Phi] = [I], \tag{8.49}$$

where : $[\Phi] = \left[\overline{\Delta}_1 \ \overline{\Delta}_2 \ ...\overline{\Delta}_n\right]$ is *n x n* matrix, called modal, where *n* is the degree of freedom of the whole system; $[\omega]$ is a diagonal matrix also called spectral, from the squares of the natural frequencies; $[I]$ is a singular matrix.

The matrix of the nodal displacements $\{\Delta\}$ can be represented as a linear combination of the vectors $\{\overline{\Delta}\}_i$ or $\{\Delta(t)\} = z_1(t)\{\overline{\Delta}\}_1 + z_2(t)\{\overline{\Delta}\}_2 + ... + z_n(t)\{\overline{\Delta}\}_n$, where $z_i(t)$ are time dependant coefficients of the division. In matrix form the division of the displacements, the velocities and the accelerations can be written like this

$$\{\Delta\} = [\Phi]\{z\}, \quad \{\dot{\Delta}\} = [\Phi]\{\dot{z}\} \text{ and } \{\ddot{\Delta}\} = [\Phi]\{\ddot{z}\}. \tag{8.50}$$

Now the equation (8.19) using the dependencies (8.49) and (8.50) and after multiplying by $[\Phi]^T$ we can write

$$\left[\omega^2\right]\{z\} + [C]_\Phi\{\dot{z}\} + \{\ddot{z}\} = [\Phi]^T [R_{вн}]. \tag{8.51}$$

In (8.51) with proportional damping according to (8.43) $[C]_\Phi$ is a diagonal matrix and $[C]_\Phi = \alpha[I] + \beta\left[\omega^2\right]$. With the so called modal damping $[C]_\Phi$ is changed with a diagonal matrix, on which the i^{th} diagonal coefficient is $C_{\Phi ii} = 2\xi_i\omega_i$, where ξ_i is the damping ratio for the i^{th} form. The modal damping has useful mathematical form and works well with small values of the damping. Because the matrixes $\left[\omega^2\right]$ and $[C]_\Phi$ are diagonal the system of differential equations (8.51) are divided. Every function $z_i(t)$ can be obtained by integrating the equation

$$\omega_i z_i(t) + 2\xi_i\omega_i\dot{z}_i(t) + \ddot{z}_i(t) = p_i \ , \quad p_i = [\Phi]_i^T \{R_{вн}\}, \tag{8.52}$$

where $\{\Phi\}_i = \{\overline{\Delta}\}_i$ as $\{\overline{\Delta}\}_i$ is scaled according to (8.33). The displacements, velocities and accelerations are obtained according to (8.50). In every equation of (8.52) p_i is a known time function. The initial values in (8.52) are obtained from the known initial values of $\{\Delta\}$ and

$\{\dot{\varDelta}\}$ in the following way. First both sides of the equation $\{\varDelta\}=[\varPhi]\{z\}$ are multiplied by $[\varPhi]^T[M]$. After that using (8.49) we can write

$$\{z(t=0)\}=[\varPhi]^T[M]\{\varDelta(t=0)\} \text{ and } \{\dot{z}(t=0)\}=[\varPhi]^T[M]\{\dot{\varDelta}(t=0)\}. \qquad (8.53)$$

After obtaining $\{z\}$ as a function of time from (8.52) $\{\varDelta\}$ is determined from $\{\varDelta\}=[\varPhi]\{z\}$.

There are many ways for time integration of (8.52). If $p_i(t)$ is partly linearly dependant on time, the exact solution for z_i can be obtained as a sum of $e^{-\xi\omega t}\sin\omega t$ and $e^{-\xi\omega t}\cos\omega t$. With more general case, more effective is the usage the direct integration method.

In many practical cases the high frequencies have a little affect on the construction and can be neglected. Then in the analysis we consider only a small amount of the low frequencies. In (8.53) only the first *m* are used to determine the displacements

$$\{\varDelta\}=\sum_{i=1}^{m}[\varPhi]_i z_i\ , \quad m<n\ . \qquad (8.54)$$

The described method is a modal method of the displacements. In the so called modal method the solution for the accelerations is obtained using modal transformation only of the inertia and viscous members of the equation (8.19).

8.8. Harmonic analysis

The goal of this analysis is to give the answer for the considered construction when the load changes in time by sinusoidal. Several loads can act at one and same time. Their change in time can be sin-phase or not, but if the frequency is the same, for analysis we can use the already considered methods. When we have loads with different frequency we should use other types of analysis, which will consider later.

When a one time harmonic loading acts it causes a transition process in the construction, which fades away in time. When we have fixed harmonic loading the construction responds with fixed motion, which has the frequency of the load. The harmonic analysis actually is the calculation of the fixed loading in the construction.

The first step of the harmonic analysis for a given construction is to calculate the natural frequencies ω_I according to (8.32) and the eigen forms of oscillation $\{\overline{\varDelta}\}_i$. After that we should obtain the functions $z_i(t)$ according to point 8.7. If we use the first few frequencies, the displacements $\{\overline{\varDelta}\}$ are determined from (8.54).

8.9. Dynamic analysis

The dynamical analysis includes consideration of the behavior of the construction when load, that changes in time, act. This analysis is also known as transient analysis. In the analysis we can use a modal method, as well as methods for direct integration of the equation of motion. The procedure when we use modal method is the following:

- the lowest values of the natural frequencies and forms are calculated;
- the modal loads p_i are calculated as a function of time;
- solve the modal equations to determine $z_i = z_i(t)$;
- according to equations (8.54) we obtain the displacements $\{\varDelta\}$;

The equations (8.52) can be solved only in several cases, for example incidentally acting load or load that changes by sinusoid.

With the methods of direct integration we don't make transformation, as it is with the modal ones. For a i^{th} moment of time t_i ($t_i = i.\varDelta t$, where $\varDelta t$ is the step of increase of the time) the equation of motion can be written

$$[K]\{\Delta\}_i + [C]\{\dot{\Delta}\}_i + [M]\{\ddot{\Delta}\}_i = \{R\}_i \,. \qquad (8.55)$$

$\{R\}_i$ is a known function in time of the load. We should find $\{\Delta\}_i$, $\{\dot{\Delta}\}_i$ and $\{\ddot{\Delta}\}_i$ in the given moment of time. After that the time increases with Δt and we observe the response of the construction in the time moment t_{i+1}. With a sequence of calculated values we can draw the diagrams of the displacements, velocities and accelerations for any degree of freedom of the construction.

8.10. Methods for direct integration of the equations of motion

The methods for obtaining the natural frequencies and forms of oscillation, that we considered earlier cost us a lot of machine time. They are sensible to be used if we have only several of the natural frequencies and forms included in the analysis. In many cases of dynamic analysis (for example when we have complex truss constructions) we have to include also the high frequency eigen forms and these methods are not so effective in this cases. More sensible is to use the methods of direct integration of (8.19) with the help of numerical procedures.

In the so called step methods of integration the matrixes $\{\Delta\}_{i+1}$, $\{\dot{\Delta}\}_{i+1}$ and $\{\ddot{\Delta}\}_{i+1}$ for some moment of time $t_{i+1} = t_i + \Delta t$ (Δt is the step of integration) are obtained using the values, that were calculated in the previous step. In comparison with the integration of (8.52), which could be numerical and where the steps for the different forms can be different, here the integration is with one and same step. And it should define exactly the oscillation, which is most important for the construction. In the cases, when the step is arbitrary bigger than the period of the high frequency oscillations the only condition is the process to be stable and the solution to be correct. These are the so called non-conditional stable methods. When we have the so called conditional stable methods of integration the step Δt should not exceed some critical value не трябва да надхвърля някаква критична стойност Δt^*.

The well known numerical methods of Adams, Rounge-Kouta, etc. are not suitable for integration of systems of higher order and that is why we have the special procedures that are shown below.

8.10.1. Method of the central differences

If the matrixes $\{\Delta\}_i$, $\{\dot{\Delta}\}_i$ and $\{\ddot{\Delta}\}_i$ are the values of the matrixes of the displacements, the velocities and the accelerations $\{\Delta(t)\}_i$, $\{\dot{\Delta}(t)\}_i$ and $\{\ddot{\Delta}(t)\}_i$ in a given moment t_i of time, for the steps $-\Delta t$ and $+\Delta t$ after calculation order with limit Δt^2 we can write the equations

$$\{\Delta\}_{i-1} = \{\Delta\}_i - \Delta t\{\dot{\Delta}\}_i + \frac{\Delta t}{2}\{\ddot{\Delta}\}_i \,, \quad \{\Delta\}_{i+1} = \{\Delta\}_i + \Delta t\{\dot{\Delta}\}_i + \frac{\Delta t}{2}\{\ddot{\Delta}\}_i \,. \qquad (8.56)$$

With summation and subtraction of the equations (8.55) we can get

$$\{\dot{\Delta}\}_i = \frac{1}{2\Delta t}\left(\{\Delta\}_{i+1} - \{\Delta\}_{i-1}\right), \; \{\ddot{\Delta}\}_i = \frac{1}{\Delta t^2}\left(\{\Delta\}_{i+1} - 2\{\Delta\}_i + \{\Delta\}_{i-1}\right). \qquad (8.57)$$

For the moment of time t_i (8.19) we can write

$$[K]\{\Delta\}_i + [C]\{\dot{\Delta}\}_i + [M]\{\ddot{\Delta}\}_i = \{R\}_i \,. \qquad (8.58)$$

If in (8.58) we put the dependencies (8.57), we can obtain the equation for $\{\Delta\}_{i+1}$

$$\left([M] + \frac{\Delta t}{2}[C]\right)\{\Delta\}_{i+1} = \{F(t)\}_i \,, \qquad (8.59)$$

where

$$\{F(t)\}_i = \Delta t^2\left(\{R\}_i - [K]\{\Delta\}_i\right) + [M](2\{\Delta\}_i - \{\Delta\}_{i-1}) + \frac{\Delta t}{2}[C]\{\Delta\}_{i-1} . \tag{8.60}$$

The equations (8.57) – (8.60) determine the procedure for the step integration of (8.19). The equations (8.57) can be considered as formulas for numerical differentiation, symmetrical for the central point $t = t_i$, from here comes the name of the method.

Because at the initial moment $t_0 = 0$ we know $\{\Delta\}_0$ and $\{\dot{\Delta}\}_0$, but not $\{\Delta\}_{i-1}$, so in the first equality of (8.56) $\{\Delta\}_{-1}$ are presented with $\{\Delta\}_0$ and we determine $\{\Delta\}_{-1} = \{\Delta\}_1 - 2\Delta t\{\Delta\}_0$, after that from the second equality we get

$$\{\ddot{\Delta}\}_0 = \frac{2}{\Delta t^2}\left(\{\Delta\}_1 - \{\Delta\}_0 - \Delta t\{\Delta\}_0\right). \tag{8.61}$$

After replacing of (8.61) in the equation

$$[K]\{\Delta\}_0 + [C]\{\dot{\Delta}\}_0 + [M]\{\ddot{\Delta}\}_0 = \{R\}_0 \tag{8.62}$$

we get

$$[M]\{\Delta\}_1 = \frac{\Delta t^2}{2}\left(\{R\}_0 - [K]\{\Delta\}_0 - [C]\{\dot{\Delta}\}_0\right) + [M]\left(\{\Delta\}_0 + \Delta t\{\Delta\}_0\right). \tag{8.63}$$

The method of the central differences is conditionally stable. If Δt is too large the calculated displacements become inexact and can grow infinitely. In order to guarantee the stableness of the solution we need to satisfy the condition

$$\Delta t < \Delta t_{кр}, \quad \Delta t_{кр} = \frac{2}{\omega_{max}} = \frac{T_{min}}{\pi} . \tag{8.64}$$

where ω_{max} is the biggest natural frequency of the construction without damping and is obtained by (8.32), while T_{min} is the minimum period of the eigen oscillations. If the model of FEA is with small dimensions, in the frequency spectrum there will be some with very high values, for which we have small oscillation period. This on the other hand will cause the steps to be much more, because of the limitation in condition (8.64).

8.10.2. Newmark's Method

The method is based on the division of $\{\Delta(t_i + \tau)\}$ and $\{\dot{\Delta}(t_i + \tau)\}$ according to the order of τ

$$\begin{aligned} \{\Delta(t_i + \tau)\} &= \{\Delta\}_i + \tau\{\dot{\Delta}\}_i + \frac{\tau^2}{2}\{\ddot{\Delta}\}_i + \alpha\tau^2\{\dddot{\Delta}\}_i \\ \{\dot{\Delta}(t_i + \tau)\} &= \{\dot{\Delta}\}_i + \tau\{\ddot{\Delta}\}_i + +\beta\tau^2\{\dddot{\Delta}\}_i \quad . \end{aligned} \tag{8.65}$$

In (8.65) α and β are coefficients, which are chosen, so the unconditional stableness of the integration process is secured. Let $\tau = \Delta t$ and we replace $\{\dddot{\Delta}\}_i$ with the approximated equality $\left(\{\ddot{\Delta}\}_{i+1} - \{\ddot{\Delta}\}_i\right)/\Delta t_i$ it can be written

$$\begin{aligned} \{\Delta\}_{i+1} &= \{\Delta\}_i + \Delta t\{\dot{\Delta}\}_i + \frac{\Delta t^2}{2}\{\ddot{\Delta}\}_i + \alpha\Delta t^2\left(\{\ddot{\Delta}\}_{i+1} - \{\ddot{\Delta}\}_i\right) \\ \{\dot{\Delta}\}_{i+1} &= \{\dot{\Delta}\}_i + \Delta t\{\ddot{\Delta}\}_i + +\beta\Delta t\left\{\{\ddot{\Delta}\}_{i+1} - \{\ddot{\Delta}\}_i\right\}_i \quad . \end{aligned} \tag{8.66}$$

Fro the first equality we obtain $\{\ddot{\Delta}\}_{i+1}$

$$\{\ddot{\Delta}\}_{i+1} = \frac{1}{\alpha\Delta t^2}\left(\{\Delta\}_{i+1} - \{\Delta\}_i\right) - \frac{1}{\alpha\Delta t}\{\dot{\Delta}\}_i + \left(1 - \frac{1}{2\alpha}\right)\{\ddot{\Delta}\}_i . \tag{8.67}$$

After replacing of (8.66) in the equation for $\{\dot{\Delta}\}_{i+1}$ we get

$$\{\dot{\Delta}\}_{i+1} = \frac{\beta}{\alpha\Delta t}\left(\{\Delta\}_{i+1} - \{\Delta\}_i\right) + \left(1 - \frac{\beta}{\alpha}\right)\{\dot{\Delta}\}_i + \frac{\Delta t}{2}\left(2 - \frac{\beta}{\alpha}\right)\{\ddot{\Delta}\}_i . \tag{8.68}$$

To obtain $\{\Delta\}_{i+1}$ the equation of motion can be written in this form

$$[K]\{\Delta\}_{i+1} + [C]\{\dot{\Delta}\}_{i+1} + [M]\{\ddot{\Delta}\}_{i+1} = \{R\}_{i+1} . \tag{8.69}$$

After replacing of (8.67) and (8.68) in the equation above, we get

$$[A]\{\Delta\}_{i+1} = \{F\}_{i+1} , \tag{8.70}$$

where

$$[A] = [K] + \frac{\beta}{\alpha\Delta t}[C] + \frac{1}{\alpha\Delta t^2}[M], \tag{8.71}$$

$$\begin{aligned}\{F\}_{i+1} = [R]_{i+1} + [M]\left[\frac{1}{\alpha\Delta t^2}\{\Delta\}_i + \frac{1}{\alpha\Delta t}\{\dot{\Delta}\}_i + \left(\frac{1}{2\alpha} - 1\right)\{\ddot{\Delta}\}_i\right] + \\ + [C]\left[\frac{\beta}{\alpha\Delta t^2}\{\Delta\}_i + \left(\frac{\beta}{\alpha} - 1\right)\{\dot{\Delta}\}_i + \frac{\Delta t}{2}\left(\frac{1}{\alpha} - 2\right)\{\ddot{\Delta}\}_i\right] .\end{aligned} \tag{8.72}$$

In order to use this method, in the initial moment $t_0 = 0$, we need to know not only the matrixes $\{\Delta\}_0$ and $\{\dot{\Delta}\}_0$, but also $\{\ddot{\Delta}\}_0$. Usually we let $\{\ddot{\Delta}\}_0 = 0$.

The right choice of the coefficients α and β can make the Newmark's method unconditionally stable. This does not mean, that we will get correct results when the step is with high value. It is observed, that we get the stableness of the method when $2\alpha \geq \alpha\beta \geq \frac{1}{2}$. More stronger criteria for unconditional stableness is the condition $0 \leq \xi < 1$ е: $\beta \geq \frac{1}{2}$ and $\alpha \geq \frac{1}{4}\left(\beta + \frac{1}{2}\right)^2$. Най-често се приема $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$.

8.11. Spectrum analysis

At the beginning of the design of a construction it is important for us to have a clear idea for the maximum values of the displacements, velocities and accelerations at a certain points, without knowing the exact moment of their occurrence. According to the methods, which we considered above, this can be done by calculating the reaction of the construction as a function of time and after that obtaining the highest values of the chosen parameters.

In the software products the so called spectrum analysis is included, with which the results are obtained relatively easy but not that correct. The simplest way to make a spectrum analysis is to use impulse load on the construction. As a reaction we obtain the maximums of every form of oscillation and after that by certain way the maximum reaction of the whole construction is obtained.

If the load is known variable as a function of time and we make a module analysis by integration of the equations (8.52) for the used in the analysis forms of oscillation. As a result we get a database for each form. In every database we search for the maximum, for example the i^{th} base has $z_{i,max}$. For a particular degree of freedom *j* the displacement Δ_{ji}, related with $z_{i,max}$ according to (8.53), can be obtained from the equation

$$\Delta_{ji} = \Phi_{ji} z_{i,max}. \tag{8.73}$$

For example $\Delta_{21} = \Phi_{21} z_{1,max}$ represent the displacement in the second degree of freedom, caused by the maximal reaction of the first form of oscillation. As a comparison the real displacement in this nodal degree of freedom is obtained from $\Delta_2 = \sum \Phi_{2i} z_i$. This sum with $z_{i,max}$ will lead to non-real physical result, because the maximums of z_i occur at different moments of time. The problem is how to determine the different values of Δ_{ji}, so we can get real values for Δ_j. One of the methods is to obtain the combination of the absolute values of Δ_{ji}, namely

$$(\Delta_j)_{max} \leq \sum_j |\Delta_{ji}|. \tag{8.74}$$

This sum can give us result bigger than the real value of Δ_j, because it is assumed, that the maximums of the forms occur at same time. More correct method is the result to be obtained from the equation

$$(\Delta_j)_{max} = \sqrt{\sum_i \Delta_{ji}^2}. \tag{8.75}$$

There are other methods, which are more complex, but they can give us more correct results.

Analogically the considerations are the same with higher degrees of freedom.