Chapter 6
PLATES

6.1. Introduction

The slabs are bodies with prismatic or cylindrical form and very small thickness compared to their bases, which are of the same order. The mid plane of the slab is the plane, which separates it into two parts. The slab's contour is the intersection line of the mid surface with the boundary surface. According to the form of the contour the slabs can be called as: circular, rectangular, triangular, elliptic, sector etc (fig. 6.1).

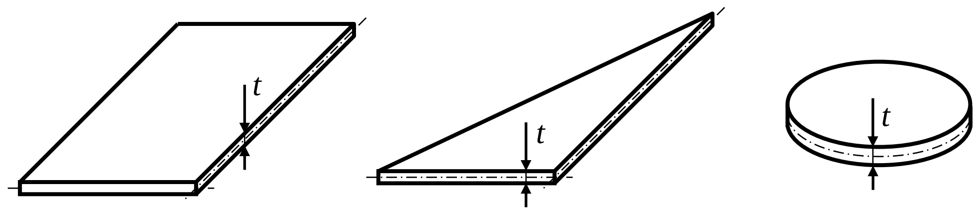


fig. 6.1

Slab's thickness may be constant, and variable too. If a is the smallest slab dimension and t is its thickness, then depending on the ratio t/a the slabs are divided into thin, if $t/a < 0{,}1$ and thick, if $t/a > 0{,}4$. Thick slabs can be treated either like massive bodies and solved like 3D problems, or additional hypothesis may be used in order to be implied an approximation method. For thick slabs the influence of the internal tangential stresses is considerable and should be taken into account.

Slab's support can be done on the contour, the internal line or in given points. It is possible a part of the contour not to be supported (free end). Slabs can be continuous (single-connected) or multiple-connected (with holes).

The loading of the slab is perpendicular to the mid surface, and the slab bends as a result. The stress state is characterized with normal and tangential stresses, which change their values with the thickness (fig. 6.2).

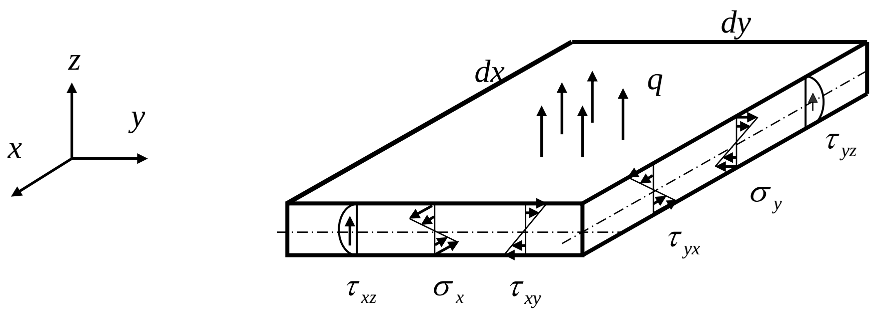


fig. 6.2

6.2. Internal forces, stress and deformations in thin plates

The theory of the thin plate is based on three hypotheses:

- kinematics hypothesis (Kirchhoff Theory), according to which the normal to the midsurface remains normal to the deformed midsurface without changing its length;
- unstretching hypothesis, according to which the points of the midsurface move only in direction of the normal (z axis);
- static hypothesis, according to which stresses in the plate with normal z are negligibly small compared to the normal stresses in slabs with normal axis x and y.

Based on this hypothesis, for slab under transverse load action $q(x,y)$ the displacements of the points are defined after its deformation (fig. 6.3).

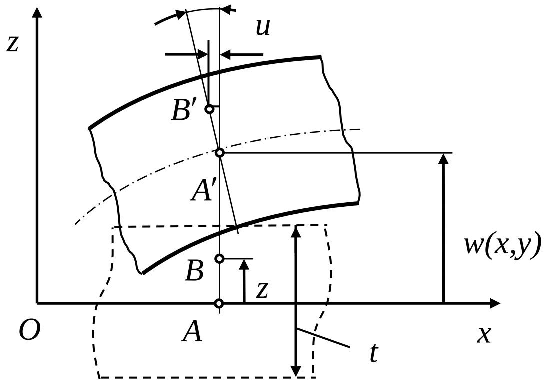


fig. 6.3

$$u(x,y,z) = -ztg\alpha = -z\frac{\partial w(x,y)}{\partial x}$$
$$v(x,y,z) = -z\frac{\partial w(x,y)}{\partial y} \qquad (6.1)$$

Because $\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}$ и $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ we can write

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2},\ \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2},\ \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \qquad (6.2)$$

where $k_x = \frac{1}{\rho_x} = -z\frac{\partial^2 w}{\partial x^2}$ - are curvature and radius of curvature for bending around axis *x*, $k_y = \frac{1}{\rho_y} = -z\frac{\partial^2 w}{\partial y^2}$ - are curvature and radius of curvature for bending around axis *y* and $k_{xy} = \frac{1}{\rho_{xy}} = -2z\frac{\partial^2 w}{\partial x \partial y}$ - curvature and radius of curvature for torsion.

The stress-strain relations according to the Hook’s law are:

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right)$$
$$\sigma_y = \frac{E}{1-\mu^2}\left(\varepsilon_y + \mu\varepsilon_x\right) \qquad (6.3)$$
$$\tau_{xy} = G\gamma_{xy} = \frac{E}{1+\mu}\gamma_{xy}$$

After substitution of the strain we get

$$\sigma_x = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) \tag{6.4}$$

$$\tau_{xy} = \tau_{yx} = -\frac{Ez}{1+\mu}\frac{\partial^2 w}{\partial x \partial y}$$

From the differential equation of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

having in mind that *X*=0 (load is in *z*) after the substitution of $\sigma_x\ u\ \tau_{xy}$ from (6.4) and integration in *z* we obtain

$$\tau_{zx} = \tau_{xz} = \frac{Ez^2}{2\left(1-\mu^2\right)}\frac{\partial}{\partial x}\nabla^2 w + f(x,y). \qquad (6.5),$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. Integration function $f(x,y)$ is defined from the condition $\tau_{zx} = 0$ for *z* = 0,5*t*. Thus we finally obtain

$$\tau_{zx} = \tau_{xz} = -\frac{E}{1-\mu^2}\left(\frac{t^2}{8} - \frac{z^2}{2}\right)\frac{\partial}{\partial x}\nabla^2 w. \tag{6.6}$$

Analogically for τ_{zy} we get

$$\tau_{zy} = \tau_{yz} = -\frac{E}{1-\mu^2}\left(\frac{t^2}{8} - \frac{z^2}{2}\right)\frac{\partial}{\partial y}\nabla^2 w. \tag{6.7}$$

Diagrams of stresses $\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}$ и τ_{yz} are shown in fig. 6.4.

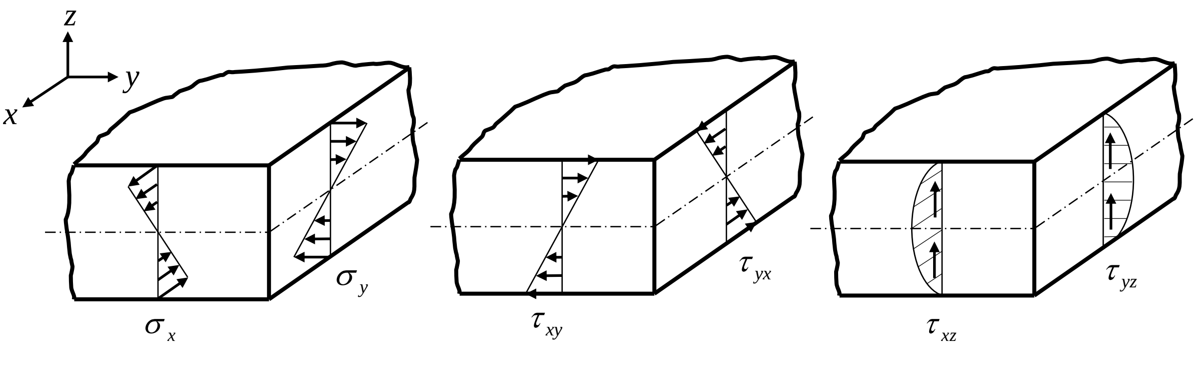


fig. 6.4

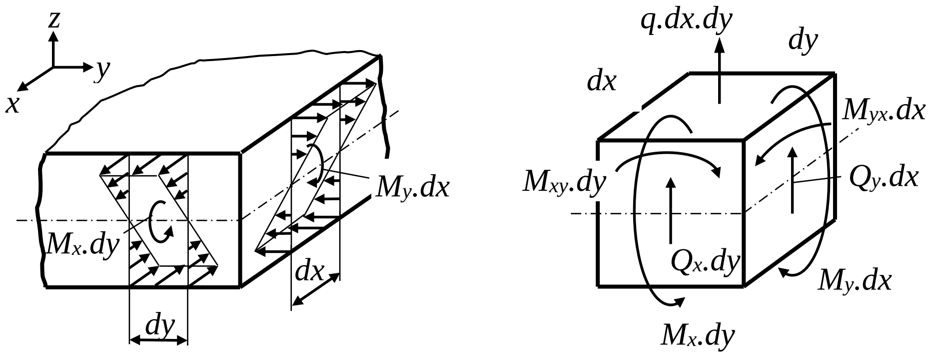
The obtained above stresses can be reduced into equivalent internal forces and moments. For example for stresses σ_x и σ_y we can write (fig. 6.5, a)

$$M_x = \int_{-t/2}^{+t/2} \sigma_x z dz, \quad M_y = \int_{-t/2}^{+t/2} \sigma_y z dz \tag{6.8}$$

Analogically are defined the rest of the internal forces

$$M_{xy} = M_{yx} = \int_{-t/2}^{t/2} \tau_{xy} z dz\,,\ Q_x = \int_{-t/2}^{t/2} \tau_{zx} dz\,,\ Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz \qquad (6.9)$$

The defined internal strains have dimensions *Nm/m* for *M* and *N/m* for *Q*.
On fig. 6.5, б internal strains are shown on a slab element loaded with distributed load $q(x, y)$.



а) б)

Fig. 6.5

Thus defined internal strains allow us to express the stresses according to the formulas

$$\sigma_x = \frac{M_x z}{t^3/12}\,,\ \sigma_y = \frac{M_y z}{t^3/12} \text{ and } \tau_{xy} = \frac{M_{xy} z}{t^3/12} \qquad (6.10)$$

6.3. FEA method for thin slabs (Kirchoff's theory)

The function of the transverse displacement of the mid surface's points $w(x, y)$ is the only independent function. The displacements of an arbitrary point from the transverse cross section of the slab are defined by (6.1).

6.3.1. Strains-displacements relations

The vector of the deformations is

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \qquad (6.11)$$

Taking into account (6.2) we may write

$$\{\varepsilon\} = -z\{\chi\} \qquad (6.12)$$

where

$$\{\chi\}^T = \left\lfloor \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad 2\frac{\partial^2 w}{\partial x \partial y} \right\rfloor \qquad (6.13)$$

6.3.2. Stress-strain relations

The vector of the stresses is $\{\sigma\}^T = \lfloor \sigma_x \ \sigma_y \ \tau_{xy} \rfloor$. According to Hook's law

$$\{\sigma\} = [E]\{\varepsilon\} \tag{6.14),}$$

where

$$[E] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \tag{6.15}$$

6.4. Quadrilateral four node uncompatible finite element

6.4.1. Displacements and strains

The element is shown on fig. 6.6, a. Each node has three degrees of freedom (fig. 6.6, б).

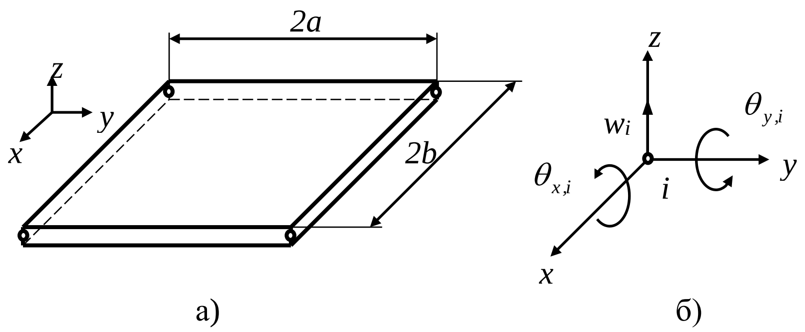


fig.6.6

The vector of the nodal displacements is

$$\{d\}^T = [\boldsymbol{d_i}\ \boldsymbol{d_j}\ \boldsymbol{d_l}\ \boldsymbol{d_m}] \tag{5.16),}$$

where

$$\{d_i\} = \begin{Bmatrix} w_i \\ \theta_{x,i} \\ \theta_{y,i} \end{Bmatrix} = \begin{Bmatrix} w_i \\ \left(\frac{\partial w}{\partial y}\right)_i \\ -\left(\frac{\partial w}{\partial x}\right)_i \end{Bmatrix} \tag{6.17}$$

The minus sign of $\left(\frac{\partial w}{\partial x}\right)_i$ is written in order to preserve the right hand rule (fig. 6.7). The vector of the nodal loads is

$$\{r\}^T = [\boldsymbol{r_i}\ \boldsymbol{r_j}\ \boldsymbol{r_l}\ \boldsymbol{r_m}] \tag{6.18),}$$

where

$$\{r_i\}^T = [F_{z,i}\ M_{x,i}\ M_{y,i}] \tag{6.19}$$

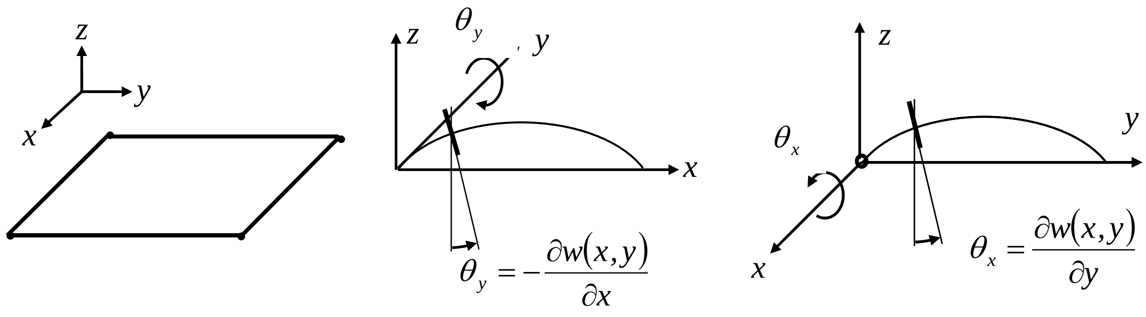


fig. 6.7

For approximation of the displacements in the element is used a partial polynomial of fourth order, because there are 12 nodal parameters. Suitable for the purpose is the polynomial

$$\begin{aligned} w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + \\ a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3 \end{aligned} \tag{6.20}$$

In direction *x=const* or *y=const* the function $w(x,y)$ will change according to a cubic law. For quadrilateral elements the borders between the elements are exactly that kind of lines (fig. 6. 8). Because of the fact that a third order polynomial is completely defined by four coefficients, two nodal displacements and two nodal rotations for nodes belonging to such line will guarantee continuity of the displacement function.

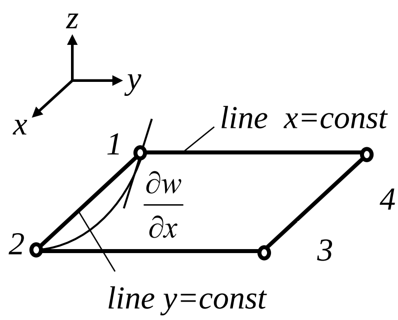


fig. 6.8

On line *x=const*, for example, the function $\frac{\partial w}{\partial x} = \theta_y = b_1 + b_2 y + b_3 y^2 + b_4 y^3$ is also a third order polynomial with four coefficients. Due to the fact that there are only two nodal parameters given along this line $\theta_{y,1}$ и $\theta_{y,2}$, the function $\frac{\partial w}{\partial x} = \theta_y$ is unconformed. Independent of the fact that it is unconformed, the element is widely used in software products, because with increase of the number of elements the solution approximates the real one.

In matrix form (6.20) we may write

$$w(x,y) = [\Phi]\{a\} \tag{6.21}$$

where $[\Phi] = \lfloor 1\ x\ y\ x^2 \ldots \rfloor$, a $\{a\}^T = \lfloor a_1\ a_2 \ldots \rfloor$ is the matrix of the polynomial coefficients.

Coefficients in (6.21) are defined from the nodal displacement equations

$$
\begin{aligned}
&w_i = a_1 + a_2 x_i + a_3 y_i + \dots \\
&\left(\frac{\partial w}{\partial y}\right)_i = \theta_{x,i} = a_3 + a_5 x + \dots \\
&\left(-\frac{\partial w}{\partial x}\right)_i = \theta_{y,i} = -a_2 - 2a_4 x - \dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots \quad i = 1,\dots,4
\end{aligned}
\tag{6.22}
$$

The equations (6.22) are 12 and can be written in matrix form as

$$\{d\} = [C]\{a\} \tag{6.23}$$

where $[C]$ is 12x12 matrix, whose elements are nodal coordinates.

From (6.23) we can define

$$\{a\} = [C]^{-1}\{d\} \tag{6.24}$$

Based on (6.21) and (6.24) we may write

$$w(x,y) = [\Phi][C]^{-1}\{d\} = [N]\{d\} \tag{6.25}$$

And then for the deformations we obtain

$$\{\varepsilon\} = -z\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = -z[D][N]\{d\} = -z[B]\{d\} \tag{6.26}$$

where $[D]^T = \left[\dfrac{\partial^2}{\partial x^2} \; \dfrac{\partial^2}{\partial y^2} \; \dfrac{\partial^2}{\partial x \partial y}\right]$.

Matrix $[B]$ can be written as

$$[B] = [\boldsymbol{B_1}\ \boldsymbol{B_2}\ \boldsymbol{B_3}\ \boldsymbol{B_4}] \tag{6.27}$$

In (6.27)

$$[B_i] = \begin{bmatrix} \dfrac{\partial^2}{\partial x^2}[N_i] \\ \dfrac{\partial^2}{\partial y^2}[N_i] \\ \dfrac{\partial^2}{\partial x \partial y}[N_i] \end{bmatrix}, \quad i = 1,\dots,4, \tag{6.28}$$

where $[N] = [\Phi][C]^{-1} = [\boldsymbol{N_1}\ \boldsymbol{N_2}\ \boldsymbol{N_3}\ \boldsymbol{N_4}]$.

From (6.26) after differentiation we obtain

$$\{\varepsilon\} = z\begin{Bmatrix} -2a_4 - 6a_7 x - 6a_8 y - 6a_{11}xy \\ -2a_6 - 2a_9 x - 6a_{10} y - 6a_{12}xy \\ -2a_5 - 4a_8 x - 4a_9 y - 6a_{11}x^2 - 6a_{12}y^2 \end{Bmatrix} \tag{6.29}$$

The relations (6.29) may be written in matrix form as

$$\{\varepsilon\} = z[Q]\{a\} \tag{6.30}$$

where

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -4x & -4y & 0 & -6x^2 & -6y^2 \end{bmatrix} \tag{6.31}$$

Using (6.24) from (6.30) we obtain

$$\{\varepsilon\} = z[Q][C]^{-1}\{d\} = z[B]\{d\} \tag{6.32}$$

from where we see that

$$[B] = [Q][C]^{-1} \tag{6.33}$$

6.4.2. Stiffness matrix of the element

It is defined from $[k] = \int [B]^T [E][B]dv$, where for *t=const* by using (6.33) we obtain

$$[k] = [[C]^{-1}]^T \left(\int_A [Q]^T [E][Q]dxdy \right) [C]^{-1} \tag{6.34}$$

When *t=const* the integral in (6.34) has an exact solution.

6.4.3. Distributed loads

If distributed load acts on the element on axis *z* *q*(x,*y*) the related nodal loads are defined from

$$\{r\} = \int_A [N]^T q(x,y)dxdy = [C]^{-1} \int_A [\Phi]^T q(x,y)dxdy \tag{6.35}$$

6.5. Quadrilateral four node finite element in local coordinate system

The element is shown in fig. 6.9.

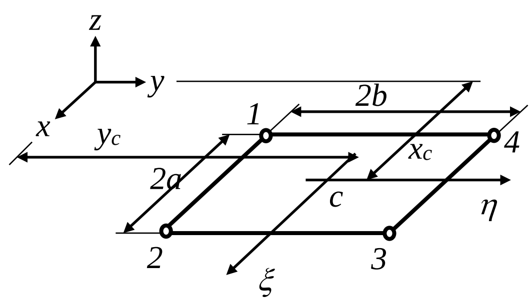


fig. 6.9

Having in mind that

$$\xi = \frac{x - x_c}{a} \text{ и } \eta = \frac{y - y_c}{b} \tag{6.36}$$

we can write

$$\frac{\partial \xi}{\partial x} = \frac{1}{a} \ , \ \frac{\partial \eta}{\partial y} = \frac{1}{b} \tag{6.37}$$

The coordinates of the nodes are: 1(-1,-1), 2(1,-1), 3(1,1), 4(-1,1). For the approximating function we can write

$$\begin{aligned} w(x,y) &= b_1 + b_2\xi + b_3\eta + b_4\xi^2 + b_5\xi\eta + b_6\eta^2 \\ &+ b_7\xi^3 + b_8\xi^2\eta + b_9\xi\eta^2 + b_{10}\eta^3 + b_{11}\xi^3\eta + b_{12}\xi\eta^3 \end{aligned} \tag{6.38}$$

or

$$w(x,y) = [P]\{b\} \quad (6.39),$$

where $[P] = \lfloor 1\, \xi\, \eta\, \xi^2 \ldots \rfloor$, and $\{b\}^T = \lfloor b_1\, b_2\, b_3 \ldots \rfloor$ is polynomial coefficient's matrix. By using nodal values of the displacements and nodal coordinates, the polynomial coefficients can be defined from the system of equations $\{d\} = [A]\{b\}$, namely

$$\{b\} = [A]^{-1}\{d\} \quad (6.40),$$

where $[A]$ is 12x12 matrix. Now the approximation function can be written as

$$w(\xi,\eta) = [P][A]^{-1}\{d\} \tag{6.41}$$

Having in mind (6.37), (6.13) is written as

$$\{\chi\}^T = \left\lfloor \frac{1}{a^2}\frac{\partial^2 w}{\partial\xi^2} \quad \frac{1}{b^2}\frac{\partial^2 w}{\partial\eta^2} \quad \frac{2}{ab}\frac{\partial^2 w}{\partial\xi\partial\eta} \right\rfloor \tag{6.42}$$

After differentiating (6.38) we obtain for the deformations:

$$\begin{aligned} \{\varepsilon\} = -z\{\chi\} = -z &\begin{bmatrix} 0 & 0 & 0 & \frac{2}{a^2} & 0 & 0 & \frac{6\xi}{a^2} & \frac{2\eta}{a^2} & 0 & 0 & \frac{6\xi\eta}{a^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{b^2} & 0 & 0 & \frac{2\xi}{b^2} & \frac{6\eta}{b^2} & 0 & \frac{6\xi\eta}{b^2} \\ 0 & 0 & 0 & 0 & \frac{2}{ab} & 0 & 0 & \frac{4\xi}{ab} & \frac{4\eta}{ab} & 0 & \frac{6\xi^2}{ab} & \frac{6\eta^2}{ab} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_{12} \end{Bmatrix} = \\ &= -z[P_\chi][A]^{-1}\{d\} = [B]\{d\} \end{aligned} \tag{6.43}$$

From (6.43) we can see that

$$[B] = -z[P_\chi][A]^{-1} \tag{6.44}$$

Stresses are defined from

$$\{\sigma\} = [E]\{\varepsilon\} = -z[E][P_\chi][A]^{-1}\{d\} \tag{6.45}$$

Stiffness matrix of the element is obtained from (2.41) where $dv = dAdz$, and $dA = abd\xi d\eta$. Then

$$\begin{aligned} \{k\} &= ab\left([A]^T\right)^{-1}\left\{ \int_{-\frac{t}{2}}^{\frac{t}{2}} \left[\int_{-1}^{1}\int_{-1}^{1} [P\chi]^T [E][P\chi] d\xi d\eta \right] z^2 dz \right\} \left[A^{-1}\right] = \\ &= ab\left([A]^T\right)^{-1}\left(\int_{-1}^{1}\int_{-1}^{1} \frac{t^3(\xi,\eta)}{12} [P\chi]^T [E][P\chi] d\xi d\eta \right)\left[A^{-1}\right]. \end{aligned} \tag{6.46}$$

6.6. Incompatible iso-parametric four node element type Kirchoff

The element is shown in fig. 6.10.

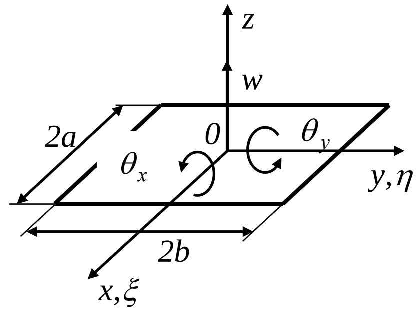


fig. 6.10

6.6.1. Shape functions

The shape functions are defined according to the formulae

$$[N_i]=\begin{bmatrix} \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)+\frac{1}{8}\xi_i\xi(1+\eta_i\eta)(1-\xi^2)+\frac{1}{8}\eta_i\eta(1+\xi_i\xi)(1-\eta^2) \\ \frac{a}{8}\xi_i(1+\xi_i\xi)(1+\eta_i\eta)(\xi^2-1) \\ \frac{a}{8}\xi_i(1+\xi_i\xi)(1+\eta_i\eta)(1-\eta^2) \end{bmatrix} \quad (6.47),$$

where ξ_i,η_i , $i=1, 2, 3, 4$ are node coordinates. Displacements approximating function is

$$w(\xi,\eta)=\sum_{i=1}^{4}N_i d_i=[N]\{d\} \quad (6.48)$$

If (6.47) is written in the form $[N_i]^T=\lfloor N_{i,1}\ N_{i,2}\ N_{i,3}\rfloor$ characteristics of these functions are:

$$N_{i,1}(\xi_i,\eta_i)=1,\ \frac{\partial N_{i,1}}{\partial\xi}(\xi_i,\eta_i)=0,\ \frac{\partial N_{i,1}}{\partial\eta}(\xi_i,\eta_i)=0$$

$$\frac{\partial N_{i,2}}{\partial x}=\frac{1}{a},\frac{\partial N_{i,2}}{\partial\xi}(\xi_i,\eta_i)=1,\frac{\partial N_{i,3}}{\partial y}=\frac{1}{b},\frac{\partial N_{i,3}}{\partial\eta}(\xi_i,\eta_i)=-1 \quad (6.49)$$

Coordinates of an arbitrary point of the element are defined from

$$x=\sum_{i=1}^{4}N_i(\xi,\eta)x_i,\ y=\sum_{i=1}^{4}N_i(\xi,\eta)y_i \quad (6.50),$$

where x_i и y_i are the coordinates of i-th node.

6.6.2. Defining the strains

According to (6.2) the strains are defined from the nodal displacements by the relations

$$\varepsilon_x=-z\frac{\partial^2 w}{\partial x^2}=-z\frac{\partial^2[N]}{\partial x^2}\{d\},\ \varepsilon_y=-z\frac{\partial^2 w}{\partial y^2}=-z\frac{\partial^2[N]}{\partial y^2}\{d\}$$

$$\gamma_{yx}=-2z\frac{\partial^2 w}{\partial x\partial y}=-2z\frac{\partial^2[N]}{\partial x\partial y}\{d\} \quad (6.51)$$

or

$$\{\varepsilon\} = -z\begin{bmatrix} \dfrac{\partial^2[N]}{\partial x^2} \\ \dfrac{\partial^2[N]}{\partial y^2} \\ 2\dfrac{\partial^2[N]}{\partial x \partial y} \end{bmatrix}\{d\} \tag{6.52}$$

From (6.52) we can see that

$$[B] = -z\begin{bmatrix} \dfrac{\partial^2[N]}{\partial x^2} \\ \dfrac{\partial^2[N]}{\partial y^2} \\ 2\dfrac{\partial^2[N]}{\partial x \partial y} \end{bmatrix} \tag{6.53}$$

$[B]$ can be expressed as $[B] = [\boldsymbol{B_1}\ \boldsymbol{B_2}\ \boldsymbol{B_3}\ \boldsymbol{B_4}]$, where

$$[B_1] = -z\begin{bmatrix} \dfrac{\partial^2[N_1]}{\partial x^2} \\ \dfrac{\partial^2[N_1]}{\partial y^2} \\ 2\dfrac{\partial^2[N_1]}{\partial x \partial y} \end{bmatrix} = \text{-z}\begin{bmatrix} \dfrac{1}{a^2}\dfrac{\partial^2[N_1]}{\partial \xi^2} \\ \dfrac{1}{b^2}\dfrac{\partial^2[N_1]}{\partial \eta^2} \\ 2\dfrac{1}{ab}\dfrac{\partial^2[N_1]}{\partial \xi \partial \eta} \end{bmatrix} \text{ etc}$$

6.6.3. Stiffness matrix of the element

It is determined from

$$[k] = \int_V [B]^T[E][B]dv = \int_V \begin{bmatrix} \boldsymbol{B_1} \\ \boldsymbol{B_2} \\ \boldsymbol{B_3} \\ \boldsymbol{B_4} \end{bmatrix}[E][\boldsymbol{B_1}\ \boldsymbol{B_2}\ \boldsymbol{B_3}\ \boldsymbol{B_4}]dV = \begin{bmatrix} k_{11}k_{12}\ldots\ldots\ldots \\ k_{21}k_{22}\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots k_{rs}\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} \tag{6.54}$$,

where typical sub matrix $[k_{rs}]$ is defined from

$$[k_{rs}] = \int_v [B_r]^T[E][B_s]dv \tag{6.55}$$

If we substitute $dv = dxdydz = abd\xi d\eta dz$ we obtain

$$[k_{rs}] = ab\int_{-1}^{1}\int_{-1}^{1}\int_{-\frac{t}{2}}^{\frac{t}{2}} z^2[B_r]^T[E][B_s]d\xi d\eta dz \tag{6.56}$$

After integrating on *z* we get

$$[k_{rs}] = ab\int_{-1}^{1}\int_{-1}^{1}\frac{t^2}{12}[B_r]^T[E][B]d\xi d\eta \tag{6.57}$$

6.7. FEA method for slabs with consideration of the strains due to tangential stresses.Mindlin theory.

This theory is based on the following assumptions:

1. Displacements along *z* axis are small .
2. The normal to the mid surface before deformation remains linear, but not obligatory normal to the mid surface after deformation.
3. The stresses, normal to the mid surface are neglegibly small .

On fig. 11 is depicted the deformation state of an element from a slab.

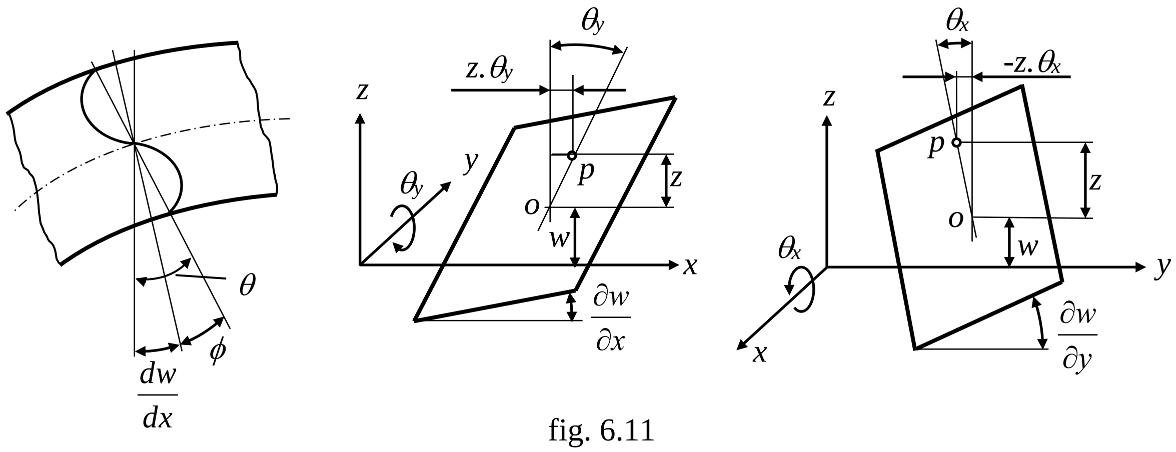


fig. 6.11

According to the assumptions and fig. 6.11 we can write

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \theta_y \ , \ \gamma_{yz} = \frac{\partial w}{\partial y} - \theta_x \tag{6.58}$$

where θ_x and θ_y are angles of rotation of the normal from the state perpendicular to the mid surface, about *x* and *z* axis respectively.

6.7.1. Strain-displacements relations

From fig. 6.11 we see, that the displacements of a point, situated on a distance *z* from the mid surface, along *x* and *y* axis are determined according to the relations:

$$u = z\theta_y \ , \ v = \text{-}z\theta_x \tag{6.59}$$

Thus for the deformations we may write

$$\varepsilon_x = \frac{\partial u}{\partial x} = z\frac{\partial \theta_y}{\partial x} \ , \ \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial \theta_x}{\partial y} \ , \ \gamma_{xy} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) \tag{6.60}$$

Relations (6.58) represent deformations from tangential internal stresses, and relations (6.60) bending deformations.

The element under consideration is based on three fields $w(x,y)$, $\theta_{\mathrm{x}}(x,y)$, $\theta_{\mathrm{y}}(x,y)$, each of them interpolated from the nodal parameters. If all fields are interpolated with one and the same polynomial, so for a finite element with *n* nodes we can write:

$$\{u\} = [N]\{d\} \tag{6.61}$$

where $\{u\}^T = \lfloor w\,\theta_x\ \theta_y \rfloor$, $\{d\}^T = \lfloor w_i\ \theta_{x,i}\ \theta_{y,i} \rfloor$ и $[N] = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix}$.

6.7.2. Strains - Displacements matrix

The deformation vector is $\{\varepsilon\}^T = \lfloor \varepsilon_x \ \varepsilon_y \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{zx} \rfloor$, and the vector of the nodal parameters for n nodes $\{d\}^T = [\boldsymbol{d_1} \ \boldsymbol{d_2} \ \boldsymbol{...d_i ...d_n}]$, where $\{d_i\}^T = \lfloor w_i \ \theta_{x,i} \ \theta_{y,i} \rfloor$. In matrix form the relation between deformations and nodal displacements is:

$$\{\varepsilon\} = [B]\{d\} \tag{6.62}$$

however $[B]$ can be represented in the form $[B] = [\boldsymbol{B_1} \ \boldsymbol{B_2} \boldsymbol{...B_i ...B_n}]$, where

$$[B_i] = z\begin{bmatrix} 0 & 0 & \dfrac{\partial N_i}{\partial x} \\ 0 & -\dfrac{\partial N_i}{\partial y} & 0 \\ 0 & -\dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & -N_i & 0 \\ \dfrac{\partial N_i}{\partial x} & 0 & N_i \end{bmatrix} \tag{6.63}$$

6.7.3. Stiffness matrix of the FE

It is determined from

$$[k] = \int_V [B]^T [E][B] dV = \begin{bmatrix} k_{11}k_{12}\ldots\ldots\ldots \\ k_{21}k_{22}\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots k_{rs}\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{bmatrix} \tag{6.64}$$

In (6.64) $[E] = \begin{bmatrix} E_{oz} & 0 \\ 0 & E_{cp} \end{bmatrix}$, where

$$[E_{oz}] = \frac{Et^2}{12(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\mu)}{2} \end{bmatrix} \tag{6.65}$$

and

$$[E_{cp}] = \frac{\alpha Et}{12(1-\mu^2)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.66}$$

In (6.66) α is a coefficient of the cross-section form, which for rectangular cross-section has value $\alpha = 6/5$. Stiffness matrix can be divided into two

$$[k_{rs,oz}] = \int_A [B_{r,oz}]^T [E_{oz}][B_{s,oz}] dxdy \tag{6.67}$$

and

$$[k_{rs,cp}] = \int_A [B_{r,cp}]^T [E_{cp}][B_{s,cp}] dxdy \tag{6.68}$$

for bending and shear respectively.

6.8. Incompatible iso-parametric element "Mindlin"

The element is shown on fig. 6.12.

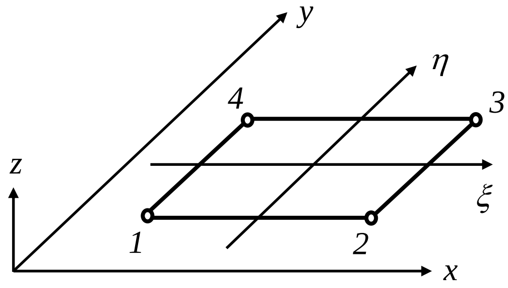


fig. 6.12

For known nodes coordinates, the coordinates of an arbitrary point in the element are determined according to the relations

$$x = \sum_{i=1}^{4} N_i(\xi,\eta)x_i \quad , \; y = \sum_{i=1}^{4} N_i(\xi,\eta)y_i \tag{6.69}$$

For this type of element are approximated three independent functions

$$w(\xi,\eta)_i = \sum_{i=1}^{4} N_i w_i$$

$$\theta_x(\xi,\eta) = \sum_{i=1}^{4} N_i \theta_{x,i} \tag{6.70}$$

$$\theta_y(\xi,\eta) = \sum_{i=1}^{4} N_i \theta_{y,i} \; .$$

Functions $N_1,...,N_4$ are of the type used in the plane problem

$$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \tag{6.71),}$$

where ξ_i и η_i are node's coordinates.

6.8.1 Strain determination

The strains due to bending are defined from

$$\varepsilon_x = z\frac{\partial\theta_y}{\partial x} = z\sum_{i=1}^{4}\frac{\partial N_i}{\partial x}\theta_{y,i}$$

$$\varepsilon_y == -z\frac{\partial\theta_x}{\partial y} = -z\sum_{i=1}^{4}\frac{\partial N_i}{\partial y}\theta_{x,i} \tag{6.72),}$$

$$\gamma_{xy} = z\left(\frac{\partial\theta_y}{\partial y} - \frac{\partial\theta_x}{\partial x}\right) = \left(\sum_{i=1}^{4}\frac{\partial N_i}{\partial y}\theta_{y,i} - \sum_{i=1}^{4}\frac{\partial N_i}{\partial x}\theta_{x,i}\right)$$

and strains due to shear from

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \theta_y = \sum_{i=1}^{4} \frac{\partial N_i}{\partial x} w_i + \sum_{i=1}^{4} N_i \theta_{y,i}$$
$$\gamma_{yz} = \frac{\partial w}{\partial y} - \theta_x = \sum_{i=1}^{4} \frac{\partial N_i}{\partial y} w_i - \sum_{i=1}^{4} N_i \theta_{x,i} \,. \tag{6.73}$$

The derivatives on x and y are determined from

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = \left[J \right]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} \tag{6.74}$$

where $\left[J \right] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$ is the Jacoby matrix.

Deformations due to bending can be combined in the matrix $\{\varepsilon_{oz}\}^T = \lfloor \varepsilon_x \; \varepsilon_y \; \gamma_{xy} \rfloor$ and according to (6.72) we may write

$$\{\varepsilon_{oz}\} = z \sum \left[B_{i,oz} \right] \{d_i\} \tag{6.75}$$

where $\{d\}^T = \lfloor w_i \; \theta_{x,i} \; \theta_{y,i} \rfloor$ and

$$\left[B_{i,oz} \right] = \begin{bmatrix} 0 & 0 & \dfrac{\partial N_i}{\partial x} \\ 0 & -\dfrac{\partial N_i}{\partial y} & 0 \\ 0 & -\dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_i}{\partial y} \end{bmatrix} \tag{6.76}$$

Deformations due to shear can be written in matrix $\{\varepsilon_{cp}\}^T = \lfloor \gamma_{xz} \; \gamma_{yz} \rfloor$ and then according to (6.73)

$$\{\varepsilon_{cp}\}^T = \sum_{i=1}^{4} \left[B_{i,cp} \right] \{d_i\} \tag{6.77}$$

where

$$\left[B_{i,cp} \right] = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 & N_i \\ \dfrac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix} \tag{6.78}$$

All deformations can be determined from

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{oz} \\ \varepsilon_{cp} \end{Bmatrix} . = \sum_{i=1}^{4} \left[B_i \right] \{d_i\} \tag{6.79}$$

where

$$[B_i] = \begin{bmatrix} z[B_{i,oг}] \\ [B_{i,cp}] \end{bmatrix} \tag{6.80}$$

6.8**.2. Stiffness matrix of the FE**

The model stiffness matrix can be represented, according to the considerations above, in parts too

$$[k_{rs}] = \lfloor k_{rs,oг} \rfloor + \lfloor k_{rs,cp} \rfloor \tag{6.81}$$

where

$$[k_{rs,oг}] = \frac{E}{1-\mu^2} \int_V z^2 [B_{r,oг}]^T [C_{oг}][B_{s,oг}] dV \tag{6.82}$$

$$[k_{rs,cp}] = G \int_V [B_{r,cp}]^T [C_{cp}][B_{s,cp}] dV \tag{6.83}$$

In (6.82) and (6.83)

$$[C_{oг}] = \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \tag{6.84}$$

$$\{C_{cp}\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.85}$$

6.9. Nodal loads

Distributed load along z axis, $q(x,y)$ is transformed into nodal loads according to

$$\{r_q\} = \int_A [N]^T q(x,y) dA \tag{6.86}$$

If there is a moment M_x, acting on an edge with direction of y-parallel, the transformation is performed according to the expression:

$$\{r_{M_x}\} = \int_l \left[\frac{\partial N}{\partial x}\right]^T M_x dy \tag{6.87}$$

Moment M_{xy}, acting on edge with destination of x-parallel, transformation is performed according to the expression:

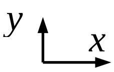
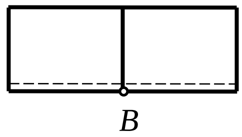
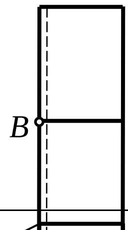
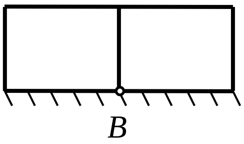
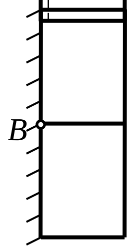
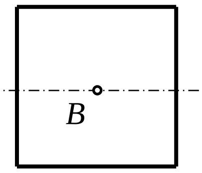
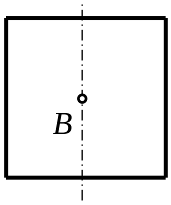
$$\{r_{M_{xy}}\} = \int_l \left[\frac{\partial N}{\partial x}\right]^T M_{xy} dx \tag{6.88}$$

In the expressions above the shape functions are related to a transverse displacement, rotation about y axis with destination of y – parallel and rotation about y axis with destination of x – parallel.

For “Kirchoff” element, according to (6.86) we obtain as nodal forces, and nodal moments too, because the shape function matrix is related to nodal displacement and two nodal rotations.

6.10. Boundary conditions

Given in table 1.

	Restriction of the displacements along line	
	Parallel to x	Parallel to y
Free support	 $w_B = 0$ $\theta_{y,B} = 0$	 $w_B = 0$ $\theta_{x,B} = 0$
Fixed support	 $w_B = 0$ $\theta_{x,B} = 0$ $\theta_{y,B} = 0$	 $w_B = 0$ $\theta_{x,B} = 0$ $\theta_{y,B} = 0$
Axis of symmetry	 $\theta_{x,B} = 0$	 $\theta_{y,B} = 0$