Chapter 4
Planar (2 dimensional) and axis-symmetrical problems. 3 dimensional problems

4.1. Introduction

In the nature all the engineering problems are 3 dimensional ones and could be considered as is. On the basis of well known from the theory of elasticity hypotheses, with acceptable errors a large amount of problems can be investigated with much simpler mathematical methods, that in modeling with FESA method leads to significant advantages too. That are problems related to the so called 2 dimensional stressed and 2 dimensional deformed state. In axis-symmetrical structures, when the material properties and the loading do not vary with the angle of rotation about the axis of symmetry, the solution of the problem for analysis of the stresses and the deformed states is similar to the shown problems in the upper sections. In the three cases we work with one and the same type of elements for constructing a mesh in the region, but the difference is in the way of integration and the volume of material, that must represent the finite element. The material volume is shown for the three cases below: - a) 2D stressed state, b) 2D deformed state c) axis-symmetrical body

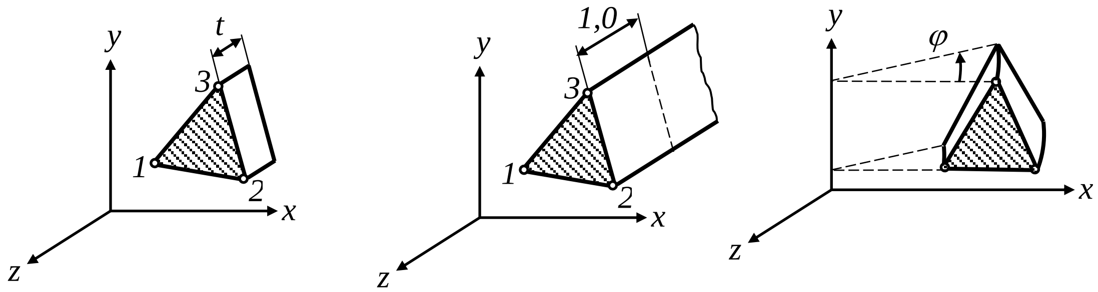


fig. 4.1

The simplest 2D finite element is the 3 node triangle element. No matter of some disadvantages, it is used intensively. Although the properties of the elements of lower order are improved ,there are developed elements from higher order too like, 3-angle 6-nodes, or 4-angle 8-nodes elements. For right shapes of the region, the last one is the perfect one, but for regions with complex shape the most suitable is the 3-angle one.

4.2. FESA method for 2D stresses and 2D deformed state analysis.3-angle 3-nodes element

The finite element is shown below

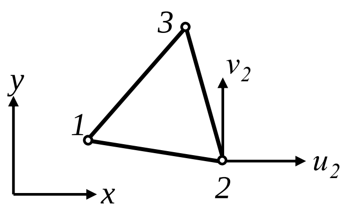


fig. 4.2

That is the simplest 2D finite element. The vector of the nodal parameters is

$$\{d\}^T = [d_1\ d_2\ d_3], \quad \{d_i\}^T = [u_i\ v_i], \quad i = 1, 2, 3 . \tag{4.1}$$

The approximation of the displacements is the FE is done with the functions

$$\begin{aligned} u(x,y) &= e_1 + e_2 x + e_3 y \\ v(x,y) &= e_4 + e_5 x + e_6 y \end{aligned} . \tag{4.2}$$

The polynomial coefficients e_1, e_2, e_3 are determined by the system of equations

$$\begin{aligned} u_1 &= e_1 + e_2 x_1 + e_3 y_1 \\ u_2 &= e_1 + e_2 x_2 + e_3 y_2 \,, \\ u_3 &= e_1 + e_2 x_3 + e_3 y_3 \end{aligned} \tag{4.3}$$

where $x_i,\ y_i\ ,\ i = 1,\ 2,\ 3$ are the nodal coordinates

After determining the coefficients, it could be written

$$u(x,y) = \frac{1}{2A}\left\{ \begin{array}{l} (a_1 + b_1 x + c_1 y)u_1 + \\ +(a_2 + b_2 x + c_2 y)u_2 + (a_3 + b_3 x + c_3 y)u_3 \end{array} \right\}, \tag{4.4}$$

where $a_1 = x_2 y_3 - x_3 y_2,\ b_1 = y_2 - y_3,\ c_1 = x_3 - x_2$, and the rest coefficients are determined by cyclic replacements of the indexes *1, 2, 3*. *2A* is the doubled area of the finite element and is defined by:

$$2A = det\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$

Similarly we can get the function *v(x,y)*

$$v(x,y) = \frac{1}{2A}\left\{ \begin{array}{l} (a_1 + b_1 x + c_1 y)v_1 + \\ +(a_2 + b_2 x + c_2 y)v_2 + (a_3 + b_3 x + c_3 y)v_3 \end{array} \right\}. \tag{4.5}$$

(4.4) and (4.5) may be written in matrix form as

$$\{u\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = [N]\{d\}, \tag{4.6}$$

where $N_i(x,y) = \frac{1}{2A}(a_i + b_i x + c_i y),\ i = 1,\ 2,\ 3$.

The deformations in the finite element are defined by the relations

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}. \tag{4.7}$$

using (4.4) and (4.5) from (4.7) we obtain

$$\{\varepsilon\}=\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}=[B]\{d\}. \quad (4.8)$$

After doing the operations on the functions N_i for the matrix $[B]$, relating the deformations and the nodal parameters,we obtain:

$$[B]=\frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}. \quad (4.9)$$

The matrix $[B]$ may be represented by blocks as

$$[\boldsymbol{B}]=[\boldsymbol{B}_1 \ \ \boldsymbol{B}_2 \ \ \boldsymbol{B}_3], \quad (4.10)$$

where $[B_i]=\frac{1}{2A}\begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix}, \ i=1,\ 2,\ 3$.

From (4.9) we can see that $[B]$, and respectively the deformations do not depend on the point's coordinates, therefore the element is known as an element with constant deformation. On the boundaries between the elements, the strains and the stresses are varying stairs-like.

Since the boundaries between the elements are straight line,then the displacements along them are defined synonymously by the nodal displacements. The element is kinematically suitable for constructing a mesh in a region with more complex shape, but is generates wrong results in using it in mesh for region with big gradient of the deformations. However, the last disadvantage could be easily avoided by making the mesh more dense.

The stiffness matrix of the element is:

$$[k]=\int_A [B]^T[E][B]tdxdy, \quad (4.11)$$

where t is the thickness of the finite element, and A is it's area. If $t=const$, since neither of the sub-integral matrices do not depends on x *and* y for the stiffness matrix we obtain

$$[k]=[B]^T[E][B]tA. \quad (4.12)$$

In block type, the stiffness matrix looks like

$$[k]=\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \quad (4.13)$$

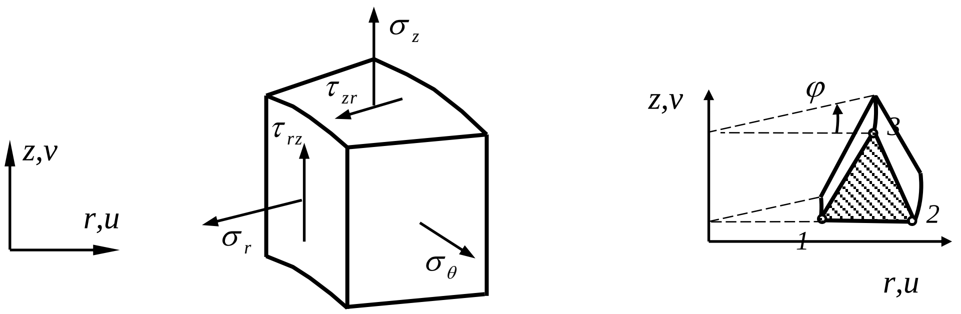
where a typical sub-matrix is derived from the equation

$$[k_{rs}]=[B_r]^T[E][B_s]tA. \quad (4.14)$$

4.3. Axis-symmetrical problem

The elements of the vectors of the stresses and the strains are respectively

$\{\sigma\}^T = [\sigma_r \ \sigma_z \ \sigma_\theta \ \tau_{rz}]$ и $\{\varepsilon\}^T = [\varepsilon_r \ \varepsilon_z \ \varepsilon_z \ \gamma_{rz}]$ (фиг. 4.3, а).



a) b)

fig. 4.3

The approximating functions for the displacements in the element (fig. 4.3, b) are

$$\begin{Bmatrix} u(r,z) \\ v(r,z) \end{Bmatrix} = \begin{bmatrix} N_1(r,z) & 0 & N_2(r,z) & 0 & N_3(r,z) & 0 \\ 0 & N_1(r,z) & 0 & N_2(r,z) & 0 & N_3(r,z) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}. \tag{4.15}$$

Now the strains can be derived from the relations

$$\begin{aligned} \varepsilon_r &= \frac{\partial u}{\partial r} = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3 \\ \varepsilon_z &= \frac{\partial v}{\partial z} = \frac{\partial N_1}{\partial z} v_1 + \frac{\partial N_2}{\partial z} v_2 + \frac{\partial N_3}{\partial z} v_3 \\ \varepsilon_\theta &= \frac{u}{r} = \frac{N_1}{r} u_1 + \frac{N_2}{r} u_2 + \frac{N_3}{r} u_3 \\ \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} = \frac{\partial N_1}{\partial z} u_1 + \frac{\partial N_2}{\partial z} u_2 + \frac{\partial N_3}{\partial z} u_3 + \frac{\partial N_1}{\partial r} v_1 + \frac{\partial N_2}{\partial r} v_2 + \frac{\partial N_3}{\partial r} v_3 \end{aligned} . \tag{4.16}$$

The relations (4.16) can be written in matrix form as follows

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = [B]\{d\}. \tag{4.17}$$

After doing the operations in (4.17) for the matrix $[B]$ we get

$$[B] = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ \frac{(a_1 + b_1 r + c_1 z)}{r} & 0 & \frac{(a_2 + b_2 r + c_2 z)}{r} & 0 & \frac{(a_3 + b_3 r + c_3 z)}{r} & 0 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}. \quad (4.18)$$

The stiffness matrix of the finite element is found by the equation

$$[k] = 2\pi \int_A [B]^T [E][B] r dr dz\ . \quad (4.19)$$

Similarly to (4.14) here we can derive the typical sub-matrix from the stiffness matrix

$$[k_{rs}] = 2\pi \int_A [B_r]^T [E][B_s] r dr dz\ . \quad (4.20)$$

4.4. 3-angle 6-nodes element

The element is shown on fig. 4.4. It has straight edges and the nodes are on the vertexes and on the middle of the edges

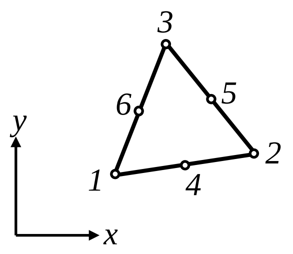


fig. 4.4

The approximating of the displacements in the FE is done by the functions

$$\begin{aligned} u(x,y) &= e_1 + e_2 x + e_3 y + e_4 x^2 + e_5 xy + e_6 y^2 \\ v(x,y) &= e_7 + e_8 x + e_9 y + e_{10} x^2 + e_{11} xy + e_{12} y^2 \end{aligned}. \quad (4.21)$$

For the strains in an element we get:

$$\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x} = e_2 + 2e_4 x + e_6 y, \quad \varepsilon_y = \frac{\partial v}{\partial y} = e_9 + e_{11} x + 2e_{12} y, \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (e_3 + e_8) + (e_5 + 2e_{10})x + (2e_6 + e_{11})y \end{aligned}. \quad (4.22)$$

It is seen that the deformations are varying linearly from *x* and *y*. The element could be called quadratic, due to the fact that field of displacements inside is a quadratic function of x and y. In modeling on pure bending with it can be found exact solutions for the drop and the stresses.

The stiffness matrix of the 3-angle 6-nodes element can be generated more easily using the so called surface or natural coordinates (fig. 4. 5).

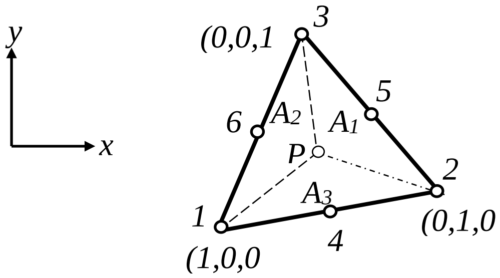


fig. 4.5

The surface coordinates of an arbitrary point *P* are defined as follows

$$\xi_1 = \frac{A_1}{A}, \quad \xi_2 = \frac{A_2}{A}, \quad \xi_3 = \frac{A_3}{A}, \tag{4.23}$$

where *A* is the surface of the triangle and $A_1+A_2+A_3=A$. Therefore

$$\xi_1 + \xi_2 + \xi_3 = 1. \tag{4.24}$$

The relations between the Decart and the surface coordinates are

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} \tag{4.25}$$

or

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_{23} & x_{32} \\ x_3 y_1 - x_1 y_3 & y_{31} & x_{13} \\ x_1 y_2 - x_2 y_1 & y_{12} & x_{21} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}, \tag{4.26}$$

and $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$, $i, j = 1, 2, 3$.

The interpolation functions of the shape are

$$\begin{matrix} N_1 = \xi_1(2\xi_1 - 1), & N_2 = \xi_2(2\xi_2 - 1), & N_3 = \xi_3(2\xi_3 - 1) \\ N_4 = 4\xi_1\xi_2, & N_5 = 4\xi_2\xi_3, & N_6 = 4\xi_3\xi_1 \end{matrix}. \tag{4.27}$$

Възловите параметри могат да се подредят във вектор както следва

$$\{d\}^T = \lfloor u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6 \ v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6 \rfloor. \tag{4.28}$$

The displacements of the element are defined by the functions

$$u = \sum_{i=1}^{6} N_i u_i, \quad v = \sum_{i=1}^{6} N_i v_i. \tag{4.29}$$

For the strain ε_x we get

$$\varepsilon_x = \frac{\partial u}{\partial x} = \sum_{i=1}^{6} \frac{\partial N_i}{\partial x} u_i, \text{ като } \frac{\partial N_i}{\partial x} = \sum_{j=1}^{3} \frac{\partial N_i}{\partial \xi_j} \frac{\partial \xi_j}{\partial x}. \tag{4.30}$$

Using (4.25), (4.27) and (4.29), on the basis of (4.30) we can write

$$\varepsilon_x = [B_x]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}, \text{ as } [B_x] = \frac{1}{2A}\begin{bmatrix} (4\xi_1 - 1)y_{23} & (4\xi_2 - 1)y_{31} & (4\xi_3 - 1)y_{12} \\ (4\xi_2 y_{23} + 4\xi_1 y_{31}) & (4\xi_3 y_{31} + 4\xi_2 y_{12}) & (4\xi_1 y_{12} + 4\xi_3 y_{23}) \end{bmatrix}. \quad (4.31)$$

Similarly for the strain ε_y we have

$$\varepsilon_y = [B_y]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}, \text{ като } [B_y] = \frac{1}{2A}\begin{bmatrix} (4\xi_1 - 1)x_{32} & (4\xi_2 - 1)x_{13} & (4\xi_3 - 1)x_{21} \\ (4\xi_2 x_{32} + 4\xi_1 x_{13}) & (4\xi_3 x_{13} + 4\xi_2 x_{21}) & (4\xi_1 x_{21} + 4\xi_3 x_{32}) \end{bmatrix}. \quad (4.32)$$

The relations strains stresses can be expressed as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{B_x} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B_y} \\ \boldsymbol{B_y} & \boldsymbol{B_x} \end{bmatrix} \{d\} = [B]\{d\} \quad (4.33)$$

Using the matrix $[B]$ we, may form the stiffness matrix of the FE too.

The 6-nodes 3-angle element in iso-parametric form is very suitable for constructiong a mesh in regions with complex shape. In the cases when the edges of the element are straight lines it's geometry is completely defined from the 3 angle points and the stiffness matrices,and the nodal loads are derived with direct integration. In transforming it from local to the global coordinate system (fig.4.6), the edges of the element can become curves, so the sub-integral functions in the stiffness matrix can not be represented with elementary polynomials and the solution can be done only in numerical way.

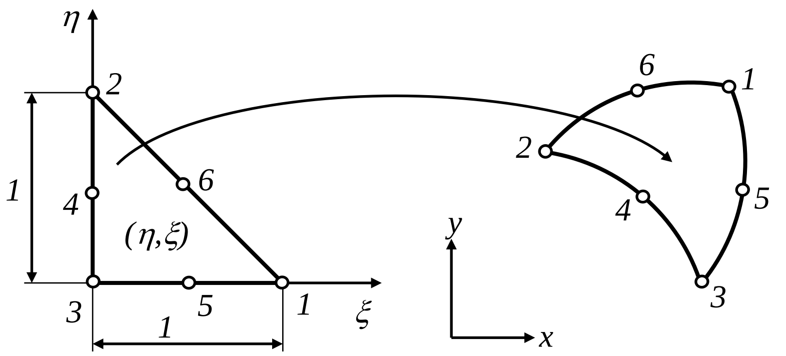


fig. 4.6

4.5. Bi-linear 4-angle element

The element is shown on fig 4.7.

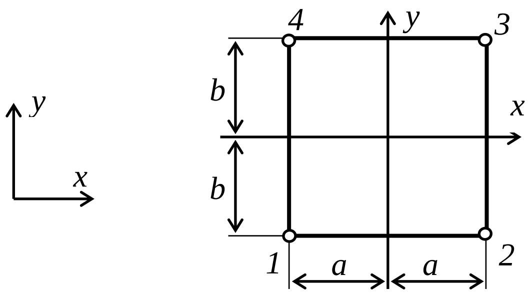


fig. 4.7

The vector of the nodal parameters is

$$\{d\}^T = \lfloor u_1\ v_1\ u_2\ v_2\ u_3\ v_3\ u_4\ v_4 \rfloor . \tag{4.34}$$

The approximation of the displacements in the FE is done with the functions

$$\begin{aligned} u &= e_1 + e_2 x + e_3 y + e_4 xy \\ v &= e_5 + e_6 x + e_7 y + e_8 xy \end{aligned} . \tag{4.35}$$

The functions of the shape are

$$\begin{aligned} N_1 &= \frac{(a-x)(b-y)}{4ab}, \quad N_2 = \frac{(a+x)(b-y)}{4ab} \\ N_3 &= \frac{(a+x)(b+y)}{4ab}, \quad N_4 = \frac{(a-x)(b+y)}{4ab} \end{aligned} . \tag{4.36}$$

On fig. 4.8 is shown the diagram of the shape function N_2

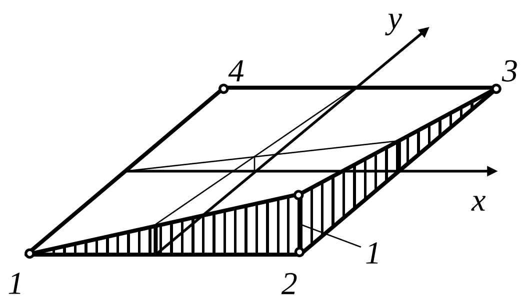


fig. 4.8

In parametric form after substitution $\xi = \frac{x}{a}$ and $\eta = \frac{y}{b}$ in (4.35) for the shape functions we obtain

$$\begin{aligned} N_1 &= \frac{1}{4}(\xi - 1)(\eta - 1) \quad N_2 = \frac{1}{4}(\xi + 1)(\eta - 1) \\ N_3 &= \frac{1}{4}(\xi + 1)(\eta + 1) \quad N_4 = \frac{1}{4}(\xi - 1)(\eta + 1). \end{aligned} \tag{4.37}$$

Thus the field of displacements is defined by the functions

$$u = \sum_{i=1}^{4} N_i u_i , \quad v = \sum_{i=1}^{4} N_i v_i . \tag{4.38}$$

For the field of displacements according to (4.35) we get:

$$\varepsilon_x = \frac{\partial u}{\partial x} = e_2 + e_4 y \,, \quad \varepsilon_y = e_7 + e_8 x \,, \quad \gamma_{xy} = (e_3 + e_6) + e_4 + e_8 y \,, \tag{4.39}$$

and according to (4.38)

$$\{\varepsilon\} = [D][N]\{d\} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots \end{bmatrix} \{d\} =$$

$$= \frac{1}{4ab} \begin{bmatrix} -(a-b) & 0 & (b-y) & 0 & \dots \\ 0 & -(a-x) & 0 & -(a+x) & \dots \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & \dots \end{bmatrix} \{d\} = [B]\{d\} \,. \tag{4.40}$$

The main properties of the element could be evaluated according to the relations (4.35) and (4.39). Since the deformation ε_x does not depend on *x*, the element can not model good the bending of the cantilever (fig. 4.9, a), where ε_x varies linearly with *x*.

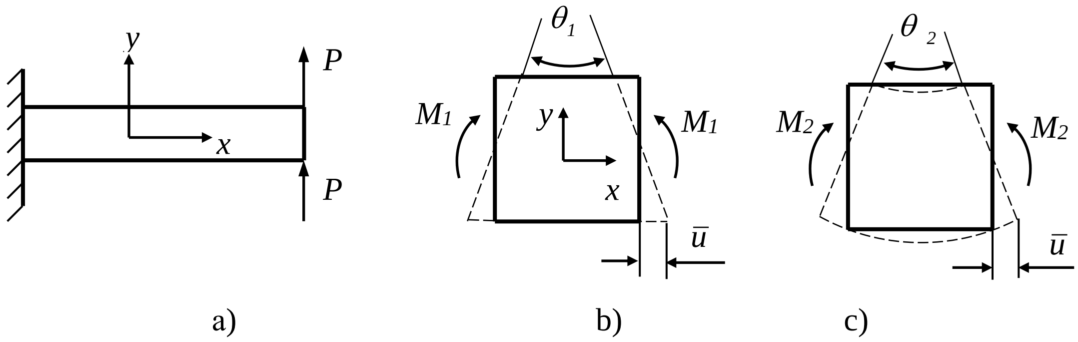


fig. 4.9

4.6. "Lock" of bi-linear 4-nodes element

The bi-linear 4-nodes element can not model good pure bending, that can be seen on the picture on fig.4.9.b). According the relations (4.37) and (4.38) in pure bending we obtain

$$u = \xi \eta \bar{u} \quad \text{and} \quad v = 0 \,, \tag{4.41}$$

therefore

$$\varepsilon_x = \eta \frac{\bar{u}}{a} \,, \quad \varepsilon_y = 0 \,, \quad \gamma_{xy} = \xi \frac{\bar{u}}{b} \,. \tag{4.42}$$

From (4.42) is easy to see, that if $\eta = \pm 1$ the upper and the lower sides of the element remains straight lines.

For pure bending ,following the Beam theory

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{y}{R} \,, \quad \varepsilon_y = \frac{\partial v}{\partial y} = -\mu \frac{y}{R} \,, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \,, \tag{4.43}$$

where R is the radius of the curve after deformation of the element,and μ is the Poisson's coefficient. After integrating of the equations (4.43) and considering the boundary conditions for the field of the displacements we obtain

$$u = \frac{xy}{R}, \quad v = \left(1-\xi^2\right)\frac{a^2}{2R} + \left(1-\eta^2\right)\frac{\mu b^2}{2R}. \tag{4.44}$$

If $\eta=\pm 1$ $\frac{a}{R}=\frac{\bar{u}}{b}$, $\frac{b}{R}=\frac{\bar{u}}{a}$ and then (43) is transformed to

$$u = \xi\eta\frac{ab}{R}, \quad v = \left(1-\xi^2\right)\frac{a\bar{u}}{2b} + \left(1-\eta^2\right)\mu\frac{b\bar{u}}{2a}. \tag{4.45}$$

According to (4.45) for the strains we get

$$\varepsilon_x = \eta\frac{\bar{u}}{a}, \quad \varepsilon_y = -\mu\eta\frac{\bar{u}}{a}, \quad \gamma_{xy} = 0. \tag{4.46}$$

The comparison of (4.46) and (4.42) shows, that the bi-linear 4-nodes element gives us wrong angle deformation, that takes part in the strain energy of the deformations. For one and the same deformations, it follows that $M_1 > M_2$. For the ratio of the defined for the two cases strain energies we get:

$$\frac{M_1}{M_2} = \frac{1}{1+\mu}\left[\frac{1}{1-\mu} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right]. \tag{4.47}$$

Equation (4.47) shows, that increasing the ratio *a/b* the element becomes more undeformable and the mesh is being LOCKED.

4.7. Improved bi-linear 4-nodes element

Basically it is the same element but with added non-nodal degrees of freedom. Each of the approximating functions of the displacements consists of 6 functions of the shape,namely:

$$\begin{aligned} u &= \sum_{i=1}^{4} N_i u_i + \left(1-\xi^2\right)g_1 + \left(1-\eta^2\right)g_2 \\ v &= \sum_{i=1}^{4} N_i v_i + \left(1-\xi^2\right)g_3 + \left(1-\eta^2\right)g_4 \end{aligned}, \tag{4.48}$$

where $\xi = \frac{x}{a}$, $\eta = \frac{y}{b}$, N_i are the shape functions from (4.36), g_1, g_2, g_3 and g_4 – non-nodal degrees of freedom, and $1-\xi^2$ and $1-\xi^2$ are non-nodal functions of the shape. Additional functions of the shape gives the possibility to model good the bending with center line on x and y axis. The shapes of the element, related to the internal degrees of freedom g_1,..g_4 are incompatible. The problems can be avoided with continuous closing uf the mesh of finite elements.

4.8. 4-nodes bi-linear iso-parametric element

The shape functions are the same like (4.36), the functions approximating the displacements are (4.38), and the coordinates of an arbitrary point from the element are defined using the functions (4.36) from

$$x = \sum_{i-1}^{4} N_i x_i \ , \quad y = \sum_{i=1}^{4} N_i y_i \ , \quad i = 1, 2, 3, 4 \ , \tag{4.49}$$

where x_i , y_i , $i = 1, 2, 3, 4$ are the coordinates of the nodes on the element

In changing the local with the global coordinate system, the element transforms itself (fig. 4.10), so the edges of the quadrangle remains straight, but the shape can not be a rectangle..

In order to obtain the matrix $[B]$ in the local coordinate system using (4.39) it is necessary to make the transformation

$$\frac{\partial N_i}{\partial x} \to \frac{\partial N_i}{\partial \xi} \text{ и } \frac{\partial N_i}{\partial y} \to \frac{\partial N_i}{\partial \eta} .$$

This can be done following the relations

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} \ , \quad \frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta} \ , \tag{4.50}$$

or

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} , \tag{4.51}$$

where $[J]$ is the matrix of Jacobi. The relationship between the global and the local derivatives comes out from the relations

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} . \tag{4.52}$$

The elementary area in the transition to the local coordinate system is transformed according to:

$$dxdy = det[J] d\xi \, d\eta \ . \tag{4.53}$$

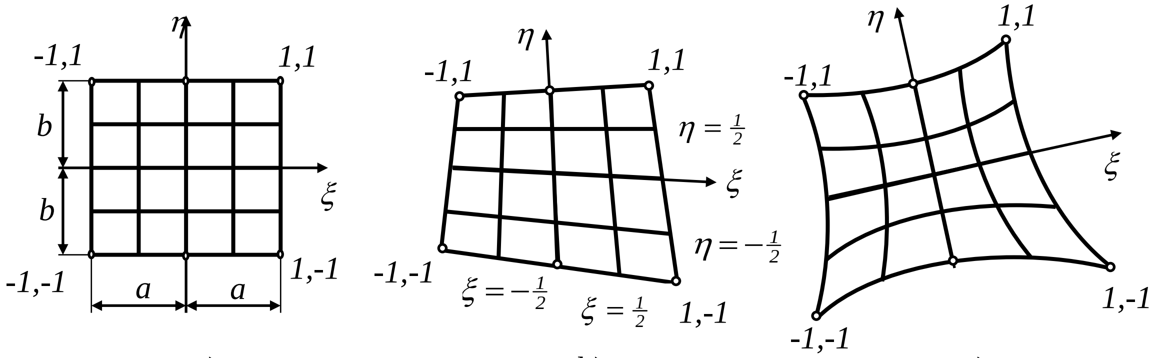
Then the stiffness matrix of the element can be written as

$$[k] = \int_v [B]^T [E][B] dv = \int_A [B]^T [E][B] t dxdy = \int_{-1}^{1}\int_{-1}^{1} [B]^T [k][B] t \, det[J] d\xi \, d\eta \ , \tag{4.54}$$

where t is the thickness of the element.

4.9. Quadratic 4-angle iso-parametric element

The element is shown in fig. 4.10 a. In transforming in the global coordinate system, it can be with straight line edges (fig. 4.10 b) or with curves (fig. 4.10 c).



a) b) c)

fig. 4.10

he current coordinates of the element are defined according to (4.49), and the displacements according to (4.38), as $i = 1, \ldots, 8$. The functions of the shape are derived from the relations

$$
\begin{aligned}
N_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \quad & N_5 &= \frac{1}{2}\left(1-\xi^2\right)(1-\eta) \\
N_2 &= -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \quad & N_6 &= \frac{1}{2}(1+\xi)\left(1-\eta^2\right) \\
N_3 &= -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \quad & N_7 &= \frac{1}{2}\left(1-\xi^2\right)(1+\eta) \\
N_4 &= -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \quad & N_8 &= \frac{1}{2}(1-\xi)\left(1-\eta^2\right) \quad .
\end{aligned}
\tag{4.55}
$$

The graph of the function N_1 is shown on fig. 4.11.

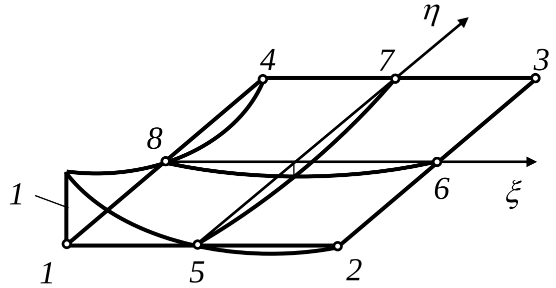


fig. 4.11

4.10. Comparative Characteristics of Two-dimensional Elements

The noted advantages and disadvantages of the considered two-dimensional finite elements could best be demonstrated by numerical analysis of the cantilever beam task, which is loaded as shown in Fig. 4.12. The beam is loaded with force *P=20* at the right end. It is equally distributed in the two nodes for models *1, 2* and *3*. In models *4, 5, 6* and *7* the force is transformed in the nodes based on distributed by quadratic parabola load with $q_c = 15$ and zero at the two ends, which corresponds to the quadratic-parabolic law for tangential stress distribution by the tangential internal strain in the cross-section. If the force unit is *N* and the dimension unit – *mm*, then the stress would be in *MPa*. The geometric boundary conditions are given so that in the end left cross-section the lateral deformation coefficient remains constant at height.

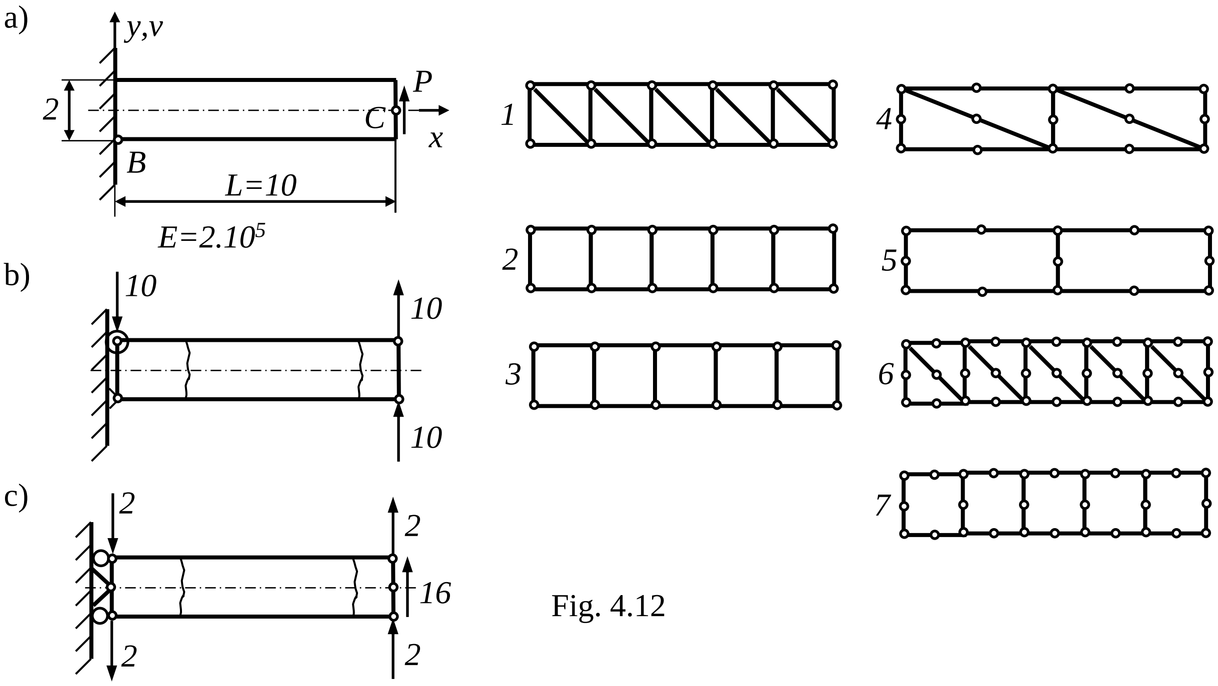


Fig. 4.12

Fig. 4.12, b and c shows the models, and Fig.12, 1-7 the finite element mesh with the solution for $\sigma_{x,B}$ in point *B* and the displacement v_c of point *C*. The solution by the beam theory, accounting for the displacement of the tangential internal strain of rectangular cross-section yields:

$$v_c = \frac{PL^3}{3EJ} + \frac{6}{5}\frac{PL}{AG} = 0{,}05 + 0{,}015 = 0{,}065\ , \quad \sigma_{x,B} = \frac{M_B}{J} z_B = \frac{200.12}{8} = 300\ .$$

As could be seen (Table 1), the triangular three node element gives very bad results. The improved bilinear rectangular element yields relatively good results that could be compared to those obtained for the cruder mesh of the higher order elements. It could be seen that the triangular six-node element and the rectangular eight-node one have considerably better capabilities to work even with cruder mesh.

Table 1

Element №	*Stresses $\sigma_{x,B}$*	*Displacement v_c*
1	*200*	*0,0346*
2	*70,1*	*0,0114*
3	*270,0*	*0,0509*
4	*253,4*	*0,0495*
5	*274,9*	*0,0504*
6	*287,7*	*0,0512*
7	*297,7*	*0,0514*

4.11. Numerical Integration. Gauss Quadrature

As mentioned earlier, to determine the stiffness matrix for the quadratic isoparametric elements with curvilinear sides, it is necessary to use numerical solution for the integrals. Most often for numerical integration is used the Gauss quadrature. It appears to be more convenient for application even in the cases with analytical integration formulae.

For one-dimensional problem the numerical solution of the integral $I = \int_{-1}^{1} \Phi(\xi)\xi$ (Fig. 4.13) could be obtained by the formula:

$$I = \sum_{i=1}^{n} W_i \Phi_i \ , \qquad (4.56)$$

where W_I is “weight” coefficient, Φ_I is the value of the function $\Phi\ (\xi)$ at specific points of the interval, called gauss points. Fig. 4.13 shows three examples with one, two and three gauss points or one-, two- and three-points gauss rule. The position of the points and the respective coefficients could be found in tabular form in the reference literature.

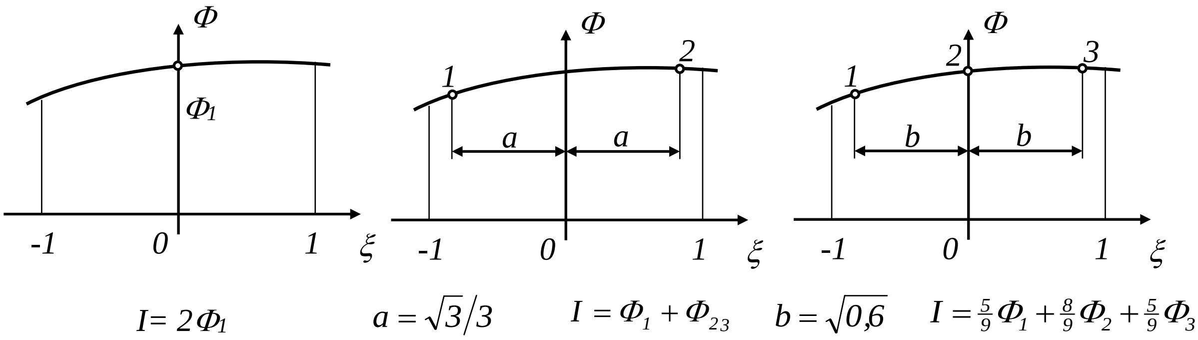


Fig. 4.13

If $\Phi\ (\xi)$ is polynomial of order $2n-1$ or less, exact solution of the integral could be obtained with n “gauss” points. Thus, exact solution of the integral having sub-integral function $\Phi\ (\xi) = e_1 + e_2\xi$ is obtained with one-point gauss rule for numerical integration, and for $\Phi\ (\xi) = e_1 + e_2\xi + e_3\xi^2$ with two-point rule. If the sub-integral function is not a polynomial, the stated above is not valid and the numerical solution with Gauss rule yields approximate solution, which could be improved through increasing the number of the gauss points.

In two-dimensional (Fig. 4.14) area $n^{\text{-the}}$ order Gauss rule contains n^2 number of points and the formula analogues to (4.55) is:

$$I = \int_{-1}^{1}\int_{-1}^{1} \Phi(\xi,\eta) d\xi\, d\eta \approx \sum_{i=1}^{n}\sum_{j=1}^{m} W_i W_j \Phi(\xi_i,\eta_j), \tag{4.57}$$

where W_I and W_j are "weight" coefficients for the two directions. Usually $n=m$, which means that the number of the gauss points in the two directions is the same. For $n=m=1$, Φ is calculated for $\xi=\eta=0$ and then $I\approx 4\Phi_1$.

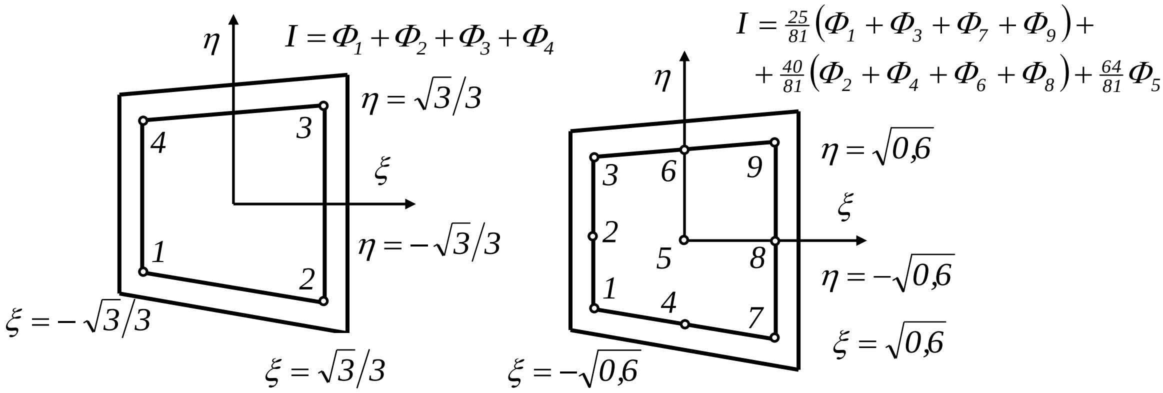


Fig. 4.14

Fig. 4.14 depicts four- and nine-points gauss rules for numerical integration. The respective integrals are:

$$I = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \tag{4.58}$$

and
$$I = \frac{25}{81}(\Phi_1 + \Phi_3 + \Phi_7 + \Phi_9) + \frac{40}{81}(\Phi_2 + \Phi_4 + \Phi_6 + \Phi_8) + \frac{64}{81}\Phi_5 . \tag{4.59}$$

In three-dimensional area the gauss quadrature of n^{-th} order contains n^3 points, and the formula for numerical integration with three "weight" coefficients for the three directions is:

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \Phi(\xi,\eta,\zeta) d\xi\, d\eta\, d\zeta \approx \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} W_i W_j W_k \Phi(\xi_i \eta_j \zeta_k). \tag{4.60}$$

4.12. Selection of Numerical Integration Order

The gauss quadrature of an order that gives the exact integrals of all the sub-matrices $\lfloor k_{ij} \rfloor$ of the stiffness matrix $[k]$ is known as "full integration". Such a "full integration" could not be provided in the cases when the elements' sides are curvilinear or the middle node is displaced. Lower order quadrature is called "reduced". The numerical integration with higher order rule does not always provide better result for the model, due to the fact that it appears to be stiffer than the real construction. The use of more gauss points when performing numerical integration leads to the occurrence of higher order members in $[k]$, which "stiffens" the model even more. Therefore, the greater accuracy of the numerical integration with more gauss points could be followed by deterioration of the accuracy of the FEA model.

Decreasing the numerical integration order leads to another effect known in FEA as instability (mechanism, kinematic form, deformed form with zero potential energy). The instability has nothing to do with the loss of rigidity and could occur when zero potential energy is obtained for all gauss points, and the respective stiffness matrix has no resistance against deformation in such a form. Examples for such instability are shown on Fig. 4.15.The integration is by one gauss point (Fig. 4.15 a), and the displacement fields for the shapes are: $u=cxy$, $v=0$ - fig. 4.15 b, $u=0, v=-cxy$ - Fig. 4.15, c and $u=cy(1-x)$, $v=cx(y-1)$ - Fig. 4.15, d (the form is known as "sand glass"). It could be seen that for each of the above deformation fields in the gauss points $u=v=0$, therefore $\varepsilon_x=\varepsilon_y=\gamma=0$. It should be noted that such a behavior is observed regardless of whether the element is rectangular or not. From the said above also follows that the depicted model cannot resist loading, which leads to such deformed forms.

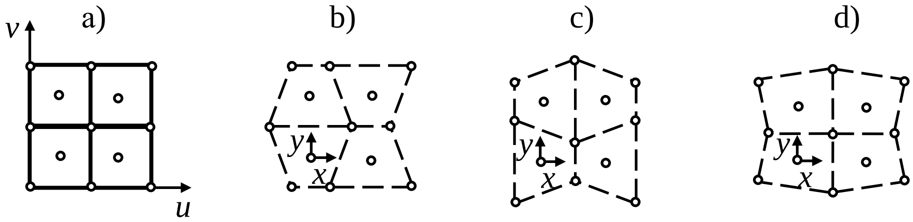


Fig. 4.15

The quadratic quadrangular element exhibits instability of "hourglass" modes when integrating the stiffness matrix by four gauss points. However, there is no way that two neighbor elements could communicate with such a deformed shape (even at inversion of the nodal displacements). Therefore a mesh of two quadratic quadrangular elements cannot exhibit instability of the type "sand-glass".

Effects, connected to the described instability could lead to entirely unacceptable results for some FEA models, like the one shown on Fig. 4.16.

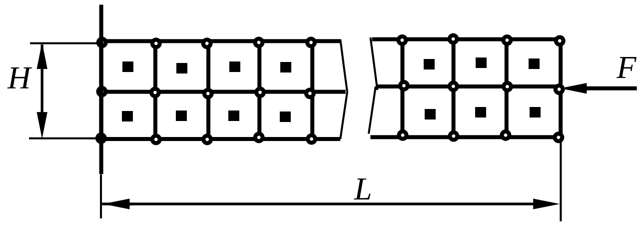


Fig. 4.16

Due to the fact that the nodes at the left end are fixed, there is no risk of instability when the elements along the length L are lower number. For mesh 2x24, however, the calculated result for the displacement of the point of the application of the force F could reach a value 500 times greater than the value calculated by the formula FL/EA due to the instability effect of the type of Fig. 4.15,b.

It should be kept in mind that usually in the commercial FEA software products, the option for numerical integration by default is for a minimal number of gauss points, which makes the occurrence of instability impossible.

4.13. Stress Calculations

It often appears that the stress calculation of the element through the relation $\{\sigma\}=[E][B]\{d\}$ becomes more accurate if the gauss points are used. For example, for a beam subject to bending and slip (fig. 4.16 a), modeled with eight-node elements, the numerical integration with four gauss points yields results (fig. 4.16 b), which show that to avoid the fake deformation γ_{xy}, the stress should be calculated for the x axis of the gauss point.

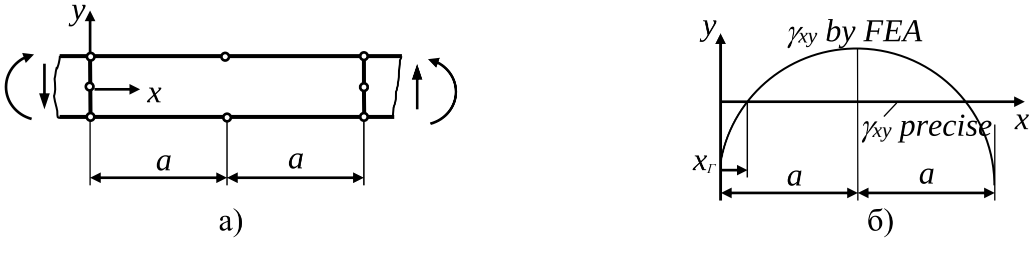


Fig. 4.17

For numerical integration in the cases of four- and eight-node plane elements the second order gauss quadrature with stress calculation in the same points is accepted. The stress in the nodes or in other points of the element are calculated by extrapolation or interpolation of the values at the gauss points, as the stress is presented in bilinear coordinate form ξ and η, or $\sigma_x = e_1 + e_2\xi + e_3\eta + e_4\xi\eta$.

4.14. FEA for Three-dimensional Problems

Many engineering problems have to be sold as three-dimensional ones. Generally speaking these are problems concerning analysis of bodies, where no limitations concerning the shape, the load, the material properties and the boundary conditions are present. The analysis includes the six possible deformations and stresses, and the field of displacements has three independent functions of the points coordinates.

As a three-dimensional problem should be considered also the problem for geometrical ax symmetrical bodies, which are not loaded and supported ax symmetrically. If only the loading is not ax symmetrical, the problem could be viewed as a combination of cases with several components of the loading. Each separate case is regarded as a two-dimensional problem, if the loading is reduced to ax symmetrical, and the final result is a sum of the results of the separate considered cases. That is the way the three-dimensional problem could be reduced to two-dimensional.

Basic problem in solving three-dimensional problems is the incredible increase of the number of equations to be solved with the increase of the elements number. Also the size of the matrix of the basic system of equations increases greatly, which leads to greater requirements to the computer configuration. The difference between the number of equations for two-dimensional problem of quadratic area with 20x20 mesh and 400 nodal points with two degrees of freedom per node and total of 800 equations could be given as an example for comparison. In this case the common matrix contains 20 nodes with about 40 variables. The equivalent three-dimensional problem is cube of 20x20x20, or 800 node points. As the degrees of freedom per node become three the total number of equations is 24000. The total matrix contains 20x20 or 400 interconnected nodes, which means 1200 equations. The problem solution time increases many times.

Most of the three-dimensional finite elements could be regarded as drawn in the two-dimensional elements space. Also their behavior could be regarded in parallel with the behavior of the two-dimensional finite elements.

4.14.1. Tetrahedron Four-node Element

That is the simplest three-dimensional finite element. It is shown on Fig. 4.18.

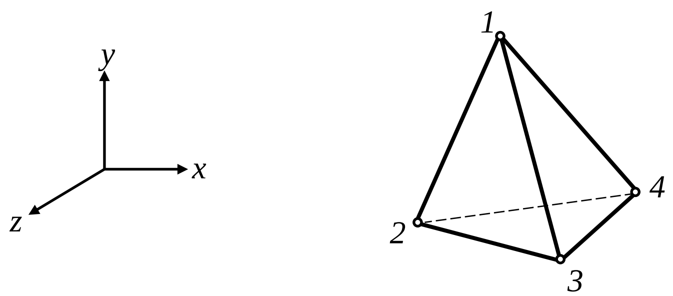


Fig. 4.18

The element has three degrees of freedom per node. The field of displacements is described by three approximating functions:

$$u = e_1 + e_2 x + e_3 y + e_4 z$$
$$v = e_5 + e_6 x + e_7 y + e_8 z \qquad (4.61)$$
$$w = e_9 + e_{10} x + e_{11} y + e_{12} z \,.$$

As with the three-node triangular element in the two-dimensional problem here the polynomial coefficients are determined through the nodal displacements and coordinates, and then for the approximation functions could be written:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ . \\ . \\ . \end{Bmatrix}, \qquad (4.62)$$

where

$$N_i = \frac{a_i + b_i x + c_i y + d_i z}{6V} \quad , i = 1, 2, 3 \,, \qquad (4.63)$$

, as $6V = det\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$ is the tetrahedron volume,and the coefficients a_1, b_1, c_1 are determined with the formulae:

$$a_1 = det\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_1 = -det\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}, \quad c_1 = -det\begin{vmatrix} x_2 & 1 & z_2 \\ x_2 & 1 & z_2 \\ x_2 & 1 & z_2 \end{vmatrix}, \quad d_1 = -det\begin{vmatrix} x_2 & y_2 & 1 \\ x_2 & y_2 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} . \quad (4.64)$$

In (4.64) $x_2, y_2, \dots$ are the nodes coordinates. The coefficients $a_2, b_2, c_2, \dots$ are determined with formulae analogues to (4.63) cyclic change of the indices *1, 2, 3, 4.*

The matrix, linking the deformations and the displacements, could be presented in block form as follows:

$$[B] = \lfloor B_i \; B_j \; B_m \; B_p \rfloor, \qquad (4.65)$$

where

$$[B_i] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix} = \frac{1}{6V} \begin{bmatrix} b_i & 0 & 0 \\ 0 & c_i & 0 \\ 0 & 0 & d_i \\ c_i & b_i & 0 \\ 0 & d_i & c_i \\ d_i & 0 & b_i \end{bmatrix}. \tag{4.66}$$

It could be seen that like in the three-node triangular element here the deformations within the element are constant. This element models badly area with big deformation gradient.

By using the matrix $[B]$ the stiffness matrix of the final element could be created

$$[k] = \int_V [B]^T [E][B] dxdydz. \tag{4.67}$$

4.14.2. Ten-node Tetrahedron Element (fig. 19, a)

The displacement field of this element could be obtained by adding six quadratic members x^2, y^2, z^2, xy, yz and zx for u, v and w in (4.61). The deformations in the element linearly depend on the coordinates. The element models accurately this pure bending. In isoparametric formulation it could have both straight walls and random shape.

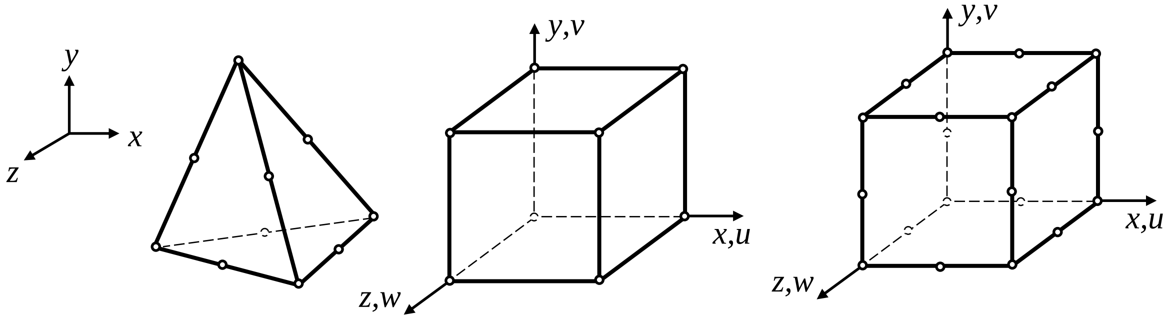


Fig. 4.19

4.14.3. Three-linear Hexahedron Element (fig. 19, b)

The element has eight nodes and a total of twenty-four degrees of freedom. The deformation field is given with the functions:

$$\begin{aligned} u &= e_1 + e_2 x + e_3 y + e_4 z + e_5 xy + e_6 yz + e_7 zx + e_8 xyz \\ v &= e_9 + e_{10} x + e_{11} y + e_{12} z + e_{13} xy + e_{14} yz + e_{15} zx + e_{16} xyz \quad . \\ w &= e_{17} + e_{18} x + e_{19} y + e_{20} z + e_{21} xy + e_{22} yz + e_{23} zx + e_{24} xyz \end{aligned} \tag{4.68}$$

Each of the above functions could be obtained from the product of three linear functions $(c_1 + c_2 x)(c_3 + c_4 y)(c_5 + c_6 z)$, where c_I, $i=1, 2, 3$ are constants.

In isoparametric formulation the element could have arbitrary form in the transition from local to global coordinate system. The displacement field is described with:

$$u = \sum_{i=1}^{8} N_i u_i \ , \quad v = \sum_{i=1}^{8} N_i v_i \ , \quad w = \sum_{i=1}^{8} N_i w_i \ . \tag{4.69}$$

The functions of form are:

$$\begin{aligned}
N_1 &= \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta), \quad N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) \\
N_3 &= \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta), \quad N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta) \\
N_5 &= \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta), \quad N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta) \\
N_7 &= \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta), \quad N_8 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta).
\end{aligned} \tag{4.70}$$

With the same functions of form are also determined the coordinates of the points of the element:

$$x = \sum_{i=1}^{8} N_i x_i \ , \quad y = \sum_{i=1}^{8} N_i y_i \ , \quad z = \sum_{i=1}^{8} N_i z_i \ , \tag{4.71}$$

where $x_1, y_1, z_1, \ldots$ are the coordinates of the nodes in the global coordinate system.

Two transformations should be done to determine the stiffness matrix:

1. The global derivatives should be expressed with the local ones, as the functions of the form are given in local coordinates.
2. The elementary volume, for which the integration is done, must be transformed in local coordinates, and the interval boundaries must be changed respectively.

For the derivatives transformation could be written:

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix}, \tag{4.72}$$

where $[J]$ is the Jacobean matrix. For $[J]$ could be written:

$$[J] = \begin{bmatrix} \sum \frac{\partial x}{\partial \xi} x_i & \sum \frac{\partial y}{\partial \xi} y_i & \sum \frac{\partial z}{\partial \xi} z_i \\ \sum \frac{\partial x}{\partial \eta} x_i & \sum \frac{\partial y}{\partial \eta} y_i & \sum \frac{\partial z}{\partial \eta} z_i \\ \sum \frac{\partial x}{\partial \zeta} x_i & \sum \frac{\partial y}{\partial \zeta} y_i & \sum \frac{\partial z}{\partial \zeta} z_i \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \ldots \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \ldots \\ \frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \ldots \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \ldots & \ldots & \ldots \end{bmatrix}. \tag{4.73}$$

In order to find the global derivatives it is necessary to obtain also the inverse matrix q, and then it could be written:

$$\left\{\begin{matrix}\frac{\partial N_i}{\partial x}\\ \frac{\partial N_i}{\partial y}\\ \frac{\partial N_i}{\partial z}\end{matrix}\right\} = \left[J\right]^{-1}\left\{\begin{matrix}\frac{\partial N_i}{\partial \xi}\\ \frac{\partial N_i}{\partial \eta}\\ \frac{\partial N_i}{\partial \zeta}\end{matrix}\right\}. \tag{4.74}$$

The volume transformation is done with the relation:

$$dxdydz = det\left|J\right| d\xi\, d\eta\, d\zeta\ . \tag{4.75}$$

The typical sub-matrix of the stiffness matrix is obtained from the integral:

$$\left[k_{rs}\right] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left[B_r\right]\left[E\right]\left[B_s\right] det\left|J\right| d\xi\, d\eta\, d\zeta\ . \tag{4.76}$$

By using the numerical solution, the matrix $\left[B\right]$ is calculated in the gauss points. The matrix $\left[J\right]$ is calculated in these points, and then its inverse matrix is calculated. After determining the derivatives $\frac{\partial N_i}{\partial x}$, $\frac{\partial N_i}{\partial y}$, $\frac{\partial N_i}{\partial z}$, from which the matrix $\left[B\right]$ is formed.

4.14.4. Quadratic Hexahedron

The element (Fig. 4.19, c) has 20 nodes and a total of 60 degrees of freedom. In isoparametric formulation, at transformation from local to global coordinate system, it could have both straight walls and arbitrary shape. If the element has rectangular walls it could model accurately linear field of the deformations. The stiffness matrix is determined with (4.67), where $\left[B\right]$ will be of size 6x60. When using numerical integration with 2x2x2 gauss quadrature instabilities of the type “sand-glass” are possible. Such a picture could be observed in sections far from the place of fixing in the example of Fig. 4.20.

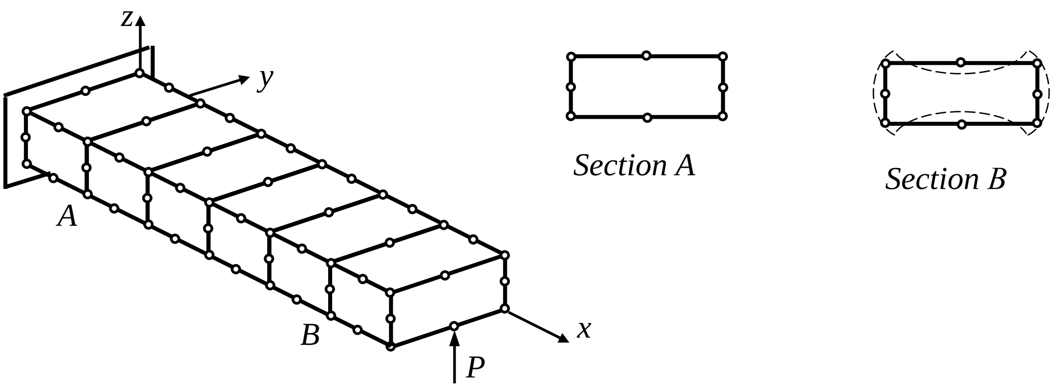


Fig. 4.20

In the software products this problem is avoided by using special 14-point rule or third order quadrature with 27 gauss points.