Chapter 3

TRUSSES AND FRAMES

3.1 Introduction

The numerical analysis of trusses and frame structures using the Finite Element Method is carried out using the standard procedure. The finite elements are the bars or beam themselves. When modeling these structures the conditions in which the beams and bars work are taken into account. In the trusses the parts are pin-jointed and may be subjected to only axial loads. The parts of a frame may be pin-jointed as well as by fixed one, and may be subjected to axial and transverse loads, as well as torques. If the elements and the loads are situated in one and the same plane, the problem is called 2D truss or frame. For so called 3D problems the structural members and the loads are arbitrarily situated in the space. Based on the Theory of elasticity exact solution for stresses and strains may be obtained only in the case of concentrated in the joints loads.

3.2 Trusses

3.2.1 Direct Method

Truss Element

Consider an elastic weightless bar with length L, modulus of elasticity E and a cross section area A (Fig.3.1). The coordinate system shown is a local one with origin at the left end. The nodes of the two-node rod element are at its two ends. It carries only axial loads and the displacements of its points are only in axial direction.

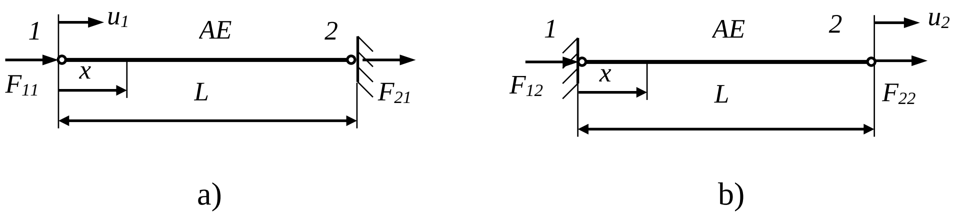


Fig. 3.1

If only node 1 of the element has displacement u_1, (Fig 3.1a), the nodal forces F_{11} and F_{21} may be determined by the formula for the absolute deformation: $\Delta L = FL/AE$, which is known from the Strength of Materials. We can write down

$$F_{11} = \frac{AE}{L}u_1 \text{ and } F_{21} = -F_{11} = -\frac{AE}{L}u_1 \qquad (3.1)$$

When only node 2 has displacement with magnitude u_2 (fig 3.1b) analogically we have

$$F_{22} = \frac{AE}{L}u_2 \text{ and } F_{12} = -F_{22} = -\frac{AE}{L}u_2 \qquad (3.2)$$

The resultant forces in the case of simultaneous displacement of the nodes 1 and 2 may be obtained by summing:

$$F_1 = F_{11} + F_{12} = \frac{AE}{L}u_1 - \frac{AE}{L}u_2 \text{ and } F_2 = F_{22} + F_{21} = \frac{AE}{L}u_2 - \frac{AE}{L}u_1 \qquad (3.3)$$

The algebraic signs of the forces and displacements are taken with positive direction being rightward. The matrix form of (3.3) is

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad (3.4)$$

or shortly

$$[k]\{d\}=\{r\} \tag{3.5}$$

where

$$\{k\}=\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.6}$$

is the stiffness matrix of the finite element; $\{d\}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ is the column matrix of the nodal displacements and $\{r\}=\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$ - column matrix of the nodal forces.

Expression (3.5) gives the relation between the nodal displacements to the nodal forces inside the finite element.

3.2.2. Variation formulation

The stiffness matrix which relates the nodal forces and the nodal displacements inside the finite element may be obtained also via expression (2.41), based on the principle of virtual displacements. For that purpose the matrix [N] of the shape function and the [B] matrix have to be determined.

The displacement function *u(x)* for the finite element may be represented as a 1-st order polynomial (Fig. 3.2,a)

$$u(x)=a_1+a_2x \tag{3.7}$$

The polynomial coefficients a_1 and a_2 are determined by the boundary conditions

$$\mathrm{u}(0)=u_1 \text{ and } u(\mathrm{L})=u_2 \tag{3.8}$$

From (3.7) we obtain

$$a_1=u_1 \text{ and } a_2=\frac{u_2-u_1}{L} \tag{3.9}$$

Now we can write for $u(x)$:

$$u(x)=u_1+\frac{u_2-u_1}{L}x=\left(1-\frac{x}{L}\right)u_1+\frac{x}{L}u_2 \tag{3.10}$$

(3.10) can be written in a shorter way:

$$\{u\}=[N]\{d\} \tag{3.11}$$

where the vector of the approximating functions $\{u\}$ has one element $u(x)$, and the matrix of the shape functions is $[N]=\lfloor N_1 \ \mathrm{N}_2 \rfloor$, where $N_1=1-\frac{x}{L}$ и $N_2=\frac{x}{L}$.

The functions N_1 and N_2 are shown graphically on Figs 3.2, *b* and 3.2, *c*

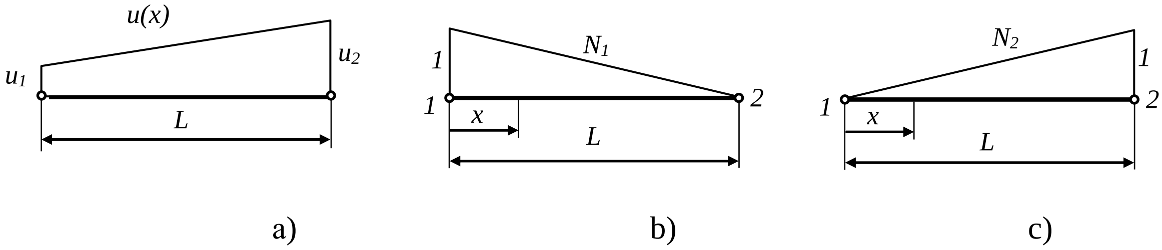


Fig. 3.2

The strain ε_x can be determined by

$$\{\varepsilon_x\} = \left\{\frac{du(x)}{dx}\right\} = \left[\frac{dN}{dx}\right]\{d\} = [B]\{d\} \tag{3.12}$$

where the matrix $\{\varepsilon\}$ has one element, ε_x, and

$$[B] = \left[\frac{dN_1}{dx}\ \frac{dN_2}{dx}\right] = \left[-\frac{1}{L}\quad \frac{1}{L}\right] \tag{3.13}$$

is the matrix which relates the strains and the nodal displacements.

For the finite element stiffness matrix from $[k] = \int_V [B]^T [E][B] dV$ we obtain

$$[k] = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \left\lfloor -\frac{1}{L}\quad \frac{1}{L} \right\rfloor AE dx = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.14}$$

Comparing (3.6) and (3.14) we see that the variation formulation in this case yields one and the same result with the Direct method of the Theory of elasticity.

With a linear approximation of the displacements (3.7), the strains (3.12) in the finite element are constant, and that's why this formulation can give a solution equal to the Theory of elasticity solution only in the case when in the rod the stresses are due to concentrated at the ends of the rod loads. With axially distributed loads a correct solution may be obtained only at the nodes, while inside the element the approximation is not correct. A solution inclining towards the exact one may be obtained by increasing the number of the finite elements. The action should be the same when $AE \neq const$.

3.2.3. Truss element in a 2D space. Truss 2D element.

In a plane truss a bar may have arbitrary orientation. In order to assemble the structure and the stiffness matrix, the bar element has to be considered in so called global coordinate system. Such a truss element is shown on Fig.3.3.

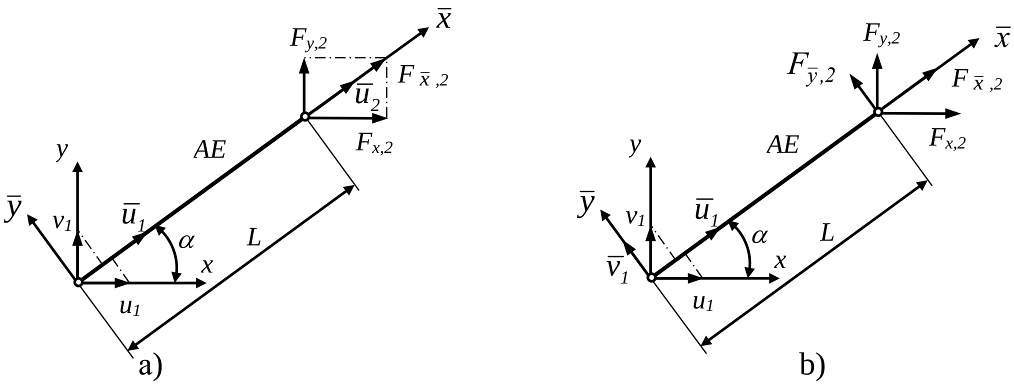


Fig. 3.3

By $\bar{x}, \bar{y}$ are designated the axes of the local coordinate system, and by x and y the axes of the global one. If the vectors of the nodal displacements in the local and global coordinate systems are respectively $\{\bar{d}\}^T = \lfloor \bar{u}_1 \; \bar{u}_2 \rfloor$ and $\{d\}^T = \lfloor u_1 \; v_1 \; u_2 \; v_2 \rfloor$ (Fig. 3.3, *a*), then the relations between the nodal parameters are:

$$\begin{aligned} \bar{u}_1 &= u_1 \cos\alpha + v_1 \sin\alpha \\ \bar{u}_2 &= u_2 \cos\alpha + v_2 \sin\alpha \end{aligned} \qquad (3.15)$$

The equation (3.15) may be rewritten in matrix form as follows:

$$\{\bar{d}\} = [T]\{d\}, \; [T] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \qquad (3.16)$$

where $[T]$ is a transformation matrix

The relations between the nodal forces of the two coordinate systems are:

$$\begin{aligned} F_{x,1} &= F_{\bar{x},1} \cos\alpha \\ F_{y,1} &= F_{\bar{x},1} \sin\alpha \\ F_{x,2} &= F_{\bar{x},2} \cos\alpha \\ F_{y,2} &= F_{\bar{x},2} \sin\alpha \end{aligned} \qquad (3.17)$$

In matrix form the equations (3.17) may be rewritten as follows:

$$\{r\} = [T]^T \{\bar{r}\} \qquad (3.18)$$

where $\{r\}^T = \left[F_{x,1} \; F_{y,1} \; F_{x,2} \; F_{y,2}\right]$ is the matrix of the nodal loads in the global coordinate system, and $\{\bar{r}\}^T = \left[F_{\bar{x},1} \; F_{\bar{x},2}\right]$ is the matrix of the nodal loads in the local coordinate system. For the local coordinate system we write

$$[\bar{k}]\{\bar{d}\} = \{\bar{r}\} \qquad (3.19)$$

where $[\bar{k}]$ is the stiffness matrix in the local coordinate system. Ssubstituting (3.16) and (3.19) in (3.18) we obtain:

$$\{r\} = [k]\{d\} \qquad (3.20)$$

where

$$\underset{4x4}{[k]} = [T]^T [\bar{k}][T] \qquad (3.21)$$

is the stiffness matrix of the finite element into the global coordinate system.

If the vectors of the nodal displacements in the local and global coordinate systems are respectively $\{\bar{d}\}^T = [\bar{u}_1\ \bar{v}_1\ \bar{u}_2\ \bar{v}_2]$ and $\{d\}^T = [u_1\ v_1\ u_2\ v_2]$ (Fig. 3.3, b), then the relation between the nodal parameters are:

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \text{ or } \{\bar{d}\} = [T]\{d\} \qquad (3.22)$$

where $c = \cos\alpha$, $s = \sin\alpha$, and $[T]$ is the transformation matrix.

The between The nodal loads of the local and global coordinate system

$$\{\bar{r}\}^T = [F_{\bar{x},1}\ F_{\bar{y},1}\ F_{\bar{x},2}\ F_{\bar{y},2}] \text{ and } \{r\}^T = [F_{x,1}\ F_{y,1}\ F_{x,2}\ F_{y,2}]$$

are related as follows

$$\{r\} = [T]^T\{\bar{r}\}\ , \qquad (3.23)$$

where $[T]^T$ is the transposed matrix from (3.22).

For the local coordinate system it may be written:

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix} = \begin{Bmatrix} F_{\bar{x},1} \\ F_{\bar{y},1} \\ F_{\bar{x},2} \\ F_{\bar{y},2} \end{Bmatrix} \text{ or } [\bar{\text{k}}]\{\bar{d}\} = \{\bar{r}\} \qquad (3.24)$$

If now (3.22) is substituted into (3.24), and (3.23) is taken into account, the following result we have:

$$[\bar{k}][T]\{d\} = \{\bar{r}\} \text{ или } [T]^{-1}[\bar{k}][T]\{d\} = \{r\}\ . \qquad (3.25)$$

$[T]$ is an orthogonal matrix, thus ($[T]^T = [T]^{-1}$). Finally we obtain

$$[T]^T[\bar{k}][T]\{d\} = \{r\} \text{ or } [k]\{d\} = \{r\} \qquad (3.26)$$

where $[k]$ is the stiffness matrix in the global coordinate system.

The stiffness matrix of the finite element may be obtained directly into the global coordinate system. On Fig.3.4 is shown a truss element in a general position with respect to the global coordinate system.

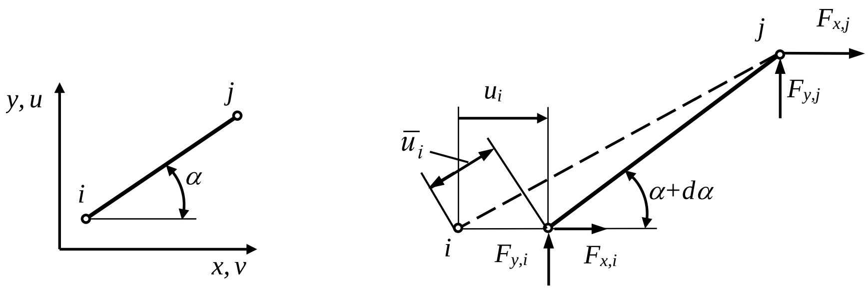


Fig. 3.4

In the global coordinate system it may be written

$$L = \left[(x_j - x_i)^2 + (y_j - y_i)^2 \right]^{\frac{1}{2}}$$
$$s = \sin\alpha = \frac{y_j - y_i}{L}\ , \ c = \cos\alpha = \frac{x_j - x_i}{L} \tag{3.27}$$

If node *i* has an axial displacement $\bar{u}_i = cu_i$ from the force $F_{\bar{x},i} = \frac{AE}{L}cu_i$, then it may be written

$$F_{\bar{x},i}c = \frac{AE}{L}c^2u_i = F_{x,i}$$
$$F_{\bar{x},i}s = \frac{AE}{l}csu_i = F_{y,i}$$
$$F_{\bar{x},j}c = \frac{AE}{L}c^2u_i = -F_{x,j} \tag{3.28}$$
$$F_{\bar{x},j}s = \frac{AE}{L}csu_i = -F_{y,j}$$

or in matrix form

$$\frac{AE}{L}\begin{Bmatrix} c^2 \\ cs \\ -c^2 \\ -cs \end{Bmatrix} u_i = \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{Bmatrix} \tag{3.29}$$

Analogical relations may also be obtained when the displacement of node *i* is v_I in the *y* axis, and also with node *j* and displacements u_j and v_j. Finally in matrix form it may be obtained

$$\frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{Bmatrix} \tag{3.30}$$

Shortly (3.30) may be written as

$$[k]\{d\} = \{r\} \tag{3.31}$$

where

$$[k] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{3.32}$$

is the stiffness matrix of the finite element in the global coordinate system.

3.2.4. Truss element in a 3D space. Truss 3D element

The element is shown in Fig. 3.5.

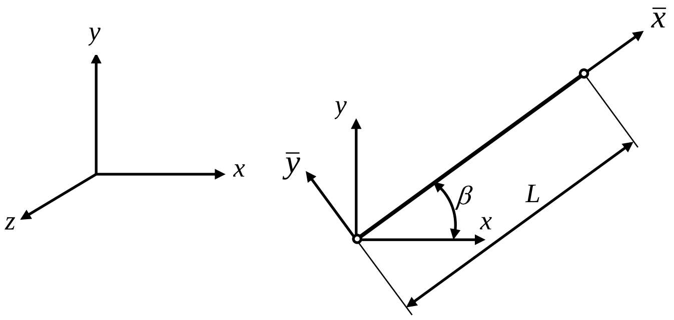


Fig. 3.5

If the vectors of the nodal displacements in the local and global coordinate systems are $\{\bar{d}\}^T = [\bar{u}_1\ \bar{v}_1\ \bar{w}_1\ \bar{u}_2\ \bar{v}_2\ \bar{w}_2]$ and $\{d\}^T = [u_1\ v_1\ w_1\ u_2\ v_2\ w_2]$, in a matrix form wehave

$$\begin{Bmatrix}\bar{u}_1\\ \bar{v}_1\\ \bar{w}_1\\ \bar{u}_2\\ \bar{v}_2\\ \bar{w}_2\end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 & 0 & 0 & 0\\ l_2 & m_2 & n_2 & 0 & 0 & 0\\ l_3 & m_3 & n_3 & 0 & 0 & 0\\ 0 & 0 & 0 & l_1 & m_1 & n_1\\ 0 & 0 & 0 & l_2 & m_2 & n_2\\ 0 & 0 & 0 & l_3 & m_3 & n_3\end{bmatrix}\begin{Bmatrix}u_1\\ v_1\\ w_1\\ u_2\\ v_2\\ w_2\end{Bmatrix} = \begin{bmatrix}[\lambda] & [0]\\ [0] & [\lambda]\end{bmatrix}\begin{Bmatrix}\{d_1\}\\ \{d_2\}\end{Bmatrix} \text{ or } \{\bar{d}\} = [T]\{d\} \qquad (3.33)$$

where $[T]$ is the transformation matrix, and $[\lambda] = \begin{bmatrix} l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}$ is the matrix of the cosines directions according to Table1.

Table 1

	x	y	z
$\bar{x}$	$l_1 = cos(\bar{x}x)$	$l_2 = cos(\bar{y}x)$	$l_3 = cos(\bar{z}x)$
$\bar{y}$	$m_1 = cos(\bar{x}y)$	$m_2 = cos(\bar{y}y)$	$m_3 = cos(\bar{z}y)$
$\bar{z}$	$n_1 = cos(\bar{x}z)$	$n_2 = cos(\bar{y}z)$	$l_3 = cos(\bar{z}z)$

For the local coordinate system it may be wtittten

$$\frac{EA}{l}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\bar{u}_1\\ \bar{v}_1\\ \bar{w}_1\\ \bar{u}_2\\ \bar{v}_2\\ \bar{w}_2\end{Bmatrix} = \begin{Bmatrix}F_{\bar{x},1}\\ F_{\bar{y},1}\\ F_{\bar{z},1}\\ F_{\bar{x},2}\\ F_{\bar{y},2}\\ F_{\bar{z},2}\end{Bmatrix} \text{ or } [\bar{k}]\{\bar{d}\} = \{\bar{r}\} \qquad (3.34)$$

where $[\bar{k}]$ is the stiffness matrix in the local coordinate system. Between the vectors of the node loads in the local and global coordinate systems $\{\bar{r}\}^T = \lfloor F_{\bar{x},1}\ F_{\bar{y},1}\ F_{\bar{z},1}\ F_{\bar{x},2}\ F_{\bar{y},2}\ F_{\bar{z},2}\rfloor$ and $\{r\}^T = \lfloor F_{x,1}\ F_{y,1}\ F_{z,1}\ F_{x,2}\ F_{y,2}\ F_{z,2}\rfloor$ the relation is

$$\{\bar{r}\} = [T]\{r\} \qquad (3.35)$$

Substituting (3.33) in (3.34) and taking into account that (3.23) it may be written

$$\left[\bar{k}\right]\left[T\right]\{d\} = \left[T\right]\{r\} \quad (3.36)$$

The matrix $[T]$ is orthogonal and finally we write

$$[T]^T\left[\bar{k}\right][T]\{d\} = \{r\} \text{ or } [k]\{d\} = \{r\} \quad (3.37)$$

where $[k]$ is the stiffness matrix in the global coordinate system.

3.2.5. Assembly of Stiffness Matrix

The assembly of stiffness matrix $[K]$ and vector of the nodal loads of the system $\{R\}$ is obtained considering the static equilibrium conditions for the nodes. In this manner it is also obtained the basec system of equations for the nodal displacements in the form:

$$[K]\{\Delta\} = \{R\} \quad (1.38)$$

3.2.6. Isoparametric bar element.

In Fig.3.6 is shown a 2-node bar element with the local *x* and the natural ζ coordinate axes, and the relation between them is: $\xi = \dfrac{x - L/2}{L/2}$.

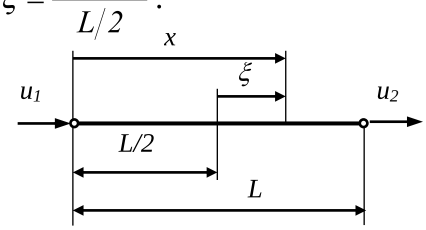


Fig. 3.6

The shape functions in the natural coordinate system are

$$N_1 = \frac{1}{2}(1-\xi)$$
$$N_2 = \frac{1}{2}(1+\xi) \quad (3.39)$$

The displacement *u* of an arbitrary point of the element is determined by the equation

$$u = N_1u_1 + N_2u_2 = \sum_{i=1}^{2} N_iu_i = [N]\{d\} \quad (3.40)$$

The *x* coordinate of an arbitrary point of the element may be analogically determined by

$$x = N_1x_1 + N_2x_2 = \sum_{i=1}^{n2} N_ix_i \quad (3.41)$$

If in (3.39) and (3.40) the same shape functions are used, the element is called isoparametric.

Since $x = x(\xi)$, we have

$$dx = \frac{\partial x}{\partial \xi} d\xi = Jd\xi \quad (3.42)$$

where $\dfrac{\partial x}{\partial \xi} = J$ is the so called Jacobian. For this element

$$J = \frac{\partial x}{\partial \xi} = \sum_{i=1}^{2} \frac{\partial N_i}{\partial \xi} x_i = -\frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_2 - x_1) = \frac{L}{2} \quad (3.43)$$

The stiffness matrix is transformed as follows

$$[k] = \int_V [B]^T [E][B] dV = \int_0^L [B(x)]^T [E][B(x)] A dx = \int_{-1}^{1} [B(\xi)]^T [E][B(\xi)] AJ d\xi \qquad (3.44)$$

In (3.44) $[B]^T$ and $[B]$ has to be transformed by substituting

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial z} = \frac{\partial N_i}{\partial \xi} J^{-1} \qquad (3.45)$$

3.3. Frames

Finite element analysis for the frames is done on the basis of the beam theory. The local coordinate system coincides with the principal axes of the cross section. When the beams of the structure are thin and long (Fig. 3.7, a) the following assumptions are used:

- The material is linear-elastic;
- Deformations are small;
- After deformation the section remains flat and perpendicular to the deformed axis of the beam;
- There is no interaction between the fibers of the beam and stresses and strains don't depend on *y*.

The angle of rotation of the section θ may be determined by

$$\theta(x) = \frac{dv(x)}{dx} \qquad (3.46)$$

where $v(x)$ is the function of the elastic line. The curvature of the beam in the *xy* plane is:

$$k = \frac{1}{R} = \frac{d^2 v(x)}{dx^2} \qquad (3.47)$$

and the strain ε_x depend linearly on *y* according to the law

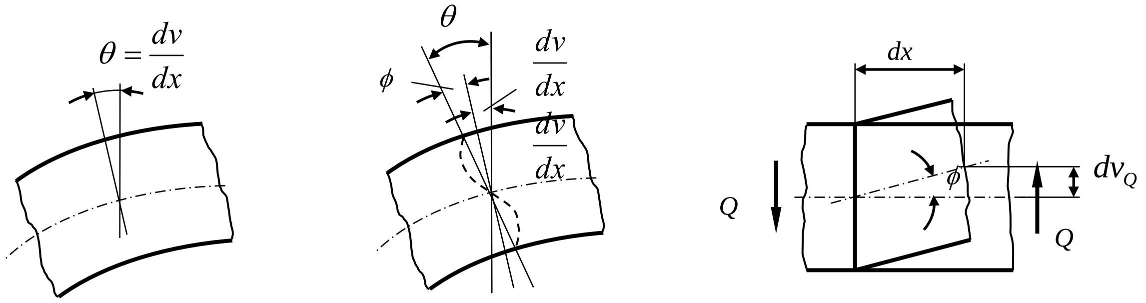
$$\varepsilon_x = \frac{y}{R} = y \frac{d^2 v(x)}{dx^2} \qquad (3.48)$$

The relation between stresses and strains is

$$\sigma_x = E\varepsilon_x \qquad (3.49)$$

With this type of beams the shear internal force may be neglected and the strain energy from the is determined by

$$U = \frac{1}{2}\sigma_x \varepsilon_x \qquad (3.50)$$



a) b) c)

Fig. 3.7

For the short and thick beams (Timoshenko type, Fig3.7, *b*) the theory is based on the additional assumption that the section after the deformation remains flat, but not necessarily perpendicular to the deformed axis of the beam. The angle of rotation of the section is determined by

$$\theta(x) = \frac{dv(x)}{dx} + \phi(x) \tag{3.51}$$

where ϕ is an additional, also called effective angle of rotation, due to the shear force in the cross section. When determining the strain energy the shear internal force *Q* must be taken into account (Fig. 3.7, *б*).

The effective angle of rotation produced by the internal force Q , $\phi(y)$ is determined according to the expression

$$\phi(y) = \frac{\tau}{G} = \frac{6Q}{Gbh}\left[\frac{1}{4} - \left(\frac{y}{h}\right)^2\right] = \frac{6Q}{GA}\left[\frac{1}{4} - \left(\frac{y}{h}\right)^2\right] \tag{3.52}$$

For the strain energy in an element with length *dx* it may be written

$$dU_Q = \frac{1}{2} Q dv_Q = \frac{1}{2}\frac{Q^2}{G\alpha A} dx \tag{3.53}$$

so

$$dv_Q = \frac{Q}{G\alpha A} dx \tag{3.54}$$

In (3.54) α is a coefficient of the shape of the cross section, *G* is the shear modulus, and *A* is the cross sectional area. Now using (3.54) for the effective angle of rotation may be written

$$\phi(x) = \frac{Q}{G\alpha A} \tag{3.55}$$

The strain energy from *Q* for an element with length *L* can be determined by

$$U_Q = \frac{1}{2}\int_0^L \frac{Q^2(x)}{G\alpha A} dx = \frac{1}{2} G\alpha A \int_0^L \phi^2 dx = \frac{1}{2} G\alpha A \int_0^L \left(\theta - \frac{dv}{dx}\right)^2 dx \tag{3.56}$$

The total strain energy is

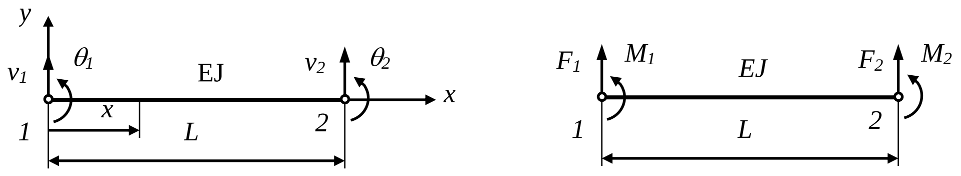
$$U = \frac{1}{2} EJ \int_0^L \left(\frac{d\theta}{dx}\right)^2 dx + \frac{1}{2} G\alpha A \int_0^L \left(\theta - \frac{dv}{dx}\right)^2 dx \tag{3.57}$$

3.3.1. FEA for thin beams. Direct method

Consider an element of constant cross sectional area, subjected to bending from transverse loads (Fig. 3.8, *a*). The element has two nodes and two degrees of freedom per node

$$\{d\}^T = \lfloor v_1 \; \theta_1 \; v_2 \; \theta_2 \rfloor \tag{3.58}$$

,where v is the lateral displacement, θ - angle of rotation. In Fig. 3.8, *b* the corresponding nodal loads are shown



а) б)

Fig. 3.8

The length of the element is *L*, the module of elasticity is *E*, and the moment of inertia about the bending axis is *J*. The corresponding loads in the nodes are shown in Fig. 8 *b*.

We suppose that the displacement of the node 1 is v_1 in the positive direction (Fig. 3.9, *a*). According to Fig. 3.9, *b* we determine the values of the loads F_{11} and M_{11}.

а б

Фиг. 3.9

Using the Castigliano’s theorem we obtain:

$$v_1 = \frac{\partial U}{\partial F_{11}} = \int \frac{M(x)}{EJ}\frac{\partial M}{\partial F_{11}}dx \text{ и } \theta_1 = \frac{\partial U}{\partial M_{11}} = \int \frac{M(x)}{EJ}\frac{\partial M}{\partial M_{11}}dx = 0 \tag{3.59}$$

,where U is the strain energy, expressed by the bending moment $M(x)$. From (3.59) we determine

$$F_{11} = \frac{12EJ}{L^3}\mathrm{v}_1\,,\ M_{11} = \frac{6EJ}{L^2}v_1 \tag{3.60}$$

From the static equilibrium conditions for of the element, we determine the loads in node 2

$$F_{21} = -\frac{12EJ}{L^3}v_1,\ M_{21} = \frac{6EJ}{L^2}v_1 \tag{3.61}$$

The first subscript of the load corresponds to the number of the node, and the second – to the number of the case in consideration.

In a similar way three more cases of nodal displacement can beconsidered and the corresponding loads may be determined. All four cases with positive directions of the loads and the displacements are shown in Fig. 3.10.

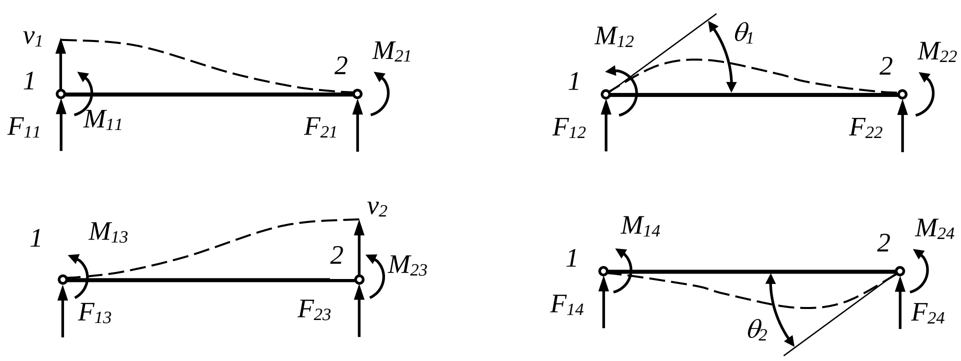


Fig. 3.10

The determined loads are:

$F_{12} = \dfrac{6EJ}{L^2}\theta_1 = -F_{22}\,,\ M_{12} = \dfrac{4EJ}{L}\theta_1,\ M_{22} = \dfrac{2EJ}{L}\theta_1$ for angle of rotation θ_1 at node 1,

$-F_{13} = \frac{12EJ}{L^2} v_2 = F_{23}$, $M_{13} = -\frac{6EJ}{L} v_2 = M_{23}$ for displacement v_2 at node 2,

$F_{14} = \frac{6EJ}{L^2}\theta_2 = F_{24}$, $M_{14} = \frac{2EJ}{L}\theta_2$, $M_{24} = \frac{4EJ}{L}\theta_2$ for angle of rotation θ_2 at node 2

For F_1 at node 1 it may be written:

$$F_1 = F_{11} + F_{12} + F_{13} + F_{14} \text{ or } F_1 = \frac{12EJ}{L^3} v_1 + \frac{6EJ}{L^2}\theta_1 - \frac{12EJ}{L^3} v_2 + \frac{6EJ}{L^2}\theta_2 \qquad (3.62)$$

In the same way F_2, M_1, M_2 can be determined. The equations can be written also in this form

$$[k]\{d\} = \{r\} \qquad (3.63)$$

, where

$$[k] = EJ\begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ & 4/L & -6/L^2 & 2/L \\ & & 12/L^3 & -6/L^2 \\ \text{symmetric} & & & 4/L \end{bmatrix} \qquad (3.64)$$

is the stiffness matrix of the finite element, $\{d\}^T = \lfloor v_1\ \theta_1\ v_2\ \theta_2 \rfloor$ is the matrix of the nodal displacements, and $\{r\} = \lfloor F_1\ M_1\ F_2\ M_2 \rfloor$ - the matrix of the nodal loads.

3.3.2. Variational formulation

The approximation of the displacements in the finite element may be obtained using the polynomials

$$v(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$
$$\theta(x) = \frac{dv(x)}{dx} = a_2 + a_3 x + 3a_4 x^2 \qquad (3.65)$$

The polynomial coefficients are determined by the boundary conditions

$$v(0) = v_1 \quad \mathrm{v(L)} = v_2$$
$$\theta(0) = \theta_1 \quad \theta(\mathrm{L}) = \theta_2 \qquad (3.66)$$

After determining of the coefficients and the corresponding rearrangements of the terms, $v(x)$ may be written in a matrix form as follows

$$v(x) = \lfloor N_1\ N_2\ N_3\ N_4 \rfloor \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = [N]\{d\} \qquad (3.67)$$

, where

$$N_1 = 1 - 3x^2/L^2 + 2x^3/L^3$$
$$N_2 = x - 2x^2/L + x^3/L^2$$
$$N_3 = 3x^2/L^2 - 2x^3/L^3 \qquad (3.68)$$
$$N_4 = -x^2/L + x^3/L^2$$

are the shape functions.

To determine the variation in the strain energy, we use the following formula

$$M = -EJ\frac{d^2v}{dx^2} \qquad (3.69)$$

The normal stress is determined via equation

$$\{\sigma\} = \frac{M}{J}y = \frac{-EJ\frac{d^2v}{dx^2}}{J}y = -E[D][N]\{d\}y = -E[B]\{d\}y \qquad (3.70$$

, where y is the coordinate of an arbitrary point of the cross section.

The strain is

$$\{\varepsilon\} = \frac{\{\sigma\}}{E} = \frac{-EJ\frac{d^2v}{dx^2}}{EJ}y = -\frac{d^2v}{dx^2}y = -[B]\{d\}y \qquad (3.71)$$

For the variation of the strain we have

$$\{\delta\varepsilon\} = -[B]\{\delta d\}y \qquad (3.72)$$

Now for the variation of the strain energy we write

$$\delta U = \int_v \{\delta\varepsilon\}^T\{\sigma\}dv = \{\delta d\}^T\int_A y^2 dA\int_0^L [B]^T E[B]dx.\{d\} = \{\delta d\}^T\int_0^L [B]^T EJ[B]dx.\{d\} \qquad (3.73)$$

If there is distributed load acting on the element the variation is

$$\delta W = \int_0^L \{\delta v\}^T\{q\}dx = \int_0^L \{\delta d\}^T[N]^T q dx \qquad (3.74)$$

Equating δU and δW, for $q = const$ we have

$$\int_0^L [B]^T EI[B]dx\{d\} = q\int_0^L [N]^T dx \qquad (3.75)$$

or

$$[k]\{d\} = q\int_0^L [N]^T dx \qquad (3.75)$$

Since $[D] = \frac{d^2}{dx^2}$ and $[N] = \lfloor N_1\ N_2\ N_3\ N_4 \rfloor$, then

$$[B] = [D][N] = \left\lfloor \frac{1}{L^3}(-6L + 12x)\ \ \frac{1}{L^2}(-4L + 6x)\ \ \frac{1}{L^3}(6L - 12x)\ \ \frac{1}{L^2}(6x - 2L) \right\rfloor \qquad (3.76)$$

Thus for the stiffness matrix of the finite element we have

$$[k] = EI\int_0^L \begin{Bmatrix} \frac{1}{L^3}(-6L+12x) \\ \frac{1}{L^2}(-4L+6x) \\ \frac{1}{L^3}(6L-12x) \\ \frac{1}{L^2}(6x-2L) \end{Bmatrix} \left\lfloor \frac{1}{L^3}(-6L+12x)\frac{1}{L^2}(-4L+6x)\frac{1}{L^3}(6L-12x)\frac{1}{L^2}(6x-2L) \right\rfloor \quad (3.77)$$

or

$$[k] = EJ\begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ & 4/L & -6/L^2 & 2/L \\ & & 12/L^3 & -6/L^2 \\ \text{symmetic} & & & 4/L \end{bmatrix} \quad (3.78$$

The result is the same as the one we obtain using the direct method. For the nodal loads e, when *q = const*, we obtain

$$\{r\} = \int_0^L [N]^T q dx = q\begin{Bmatrix} L/2 \\ L^2/12 \\ L/2 \\ -L^2/12 \end{Bmatrix} \quad (3.79$$

The curvature of the beam is

$$\frac{d^2v}{dx^2} = \left\lfloor \frac{d^2}{dx^2}N \right\rfloor \{d\} = [B]\{d\} \quad (3.80)$$

The stress in the element is determined by the beam theory, as the bending moment is determined from

$$M = EJ\frac{d^2v}{dx^2} = EJ[B]\{d\} \quad (3.81)$$

The bending moment in (3.48) depends linearly on *x*, because the elements of $[B]$ also depend linearly on *x*. Thus, we obtain exact solution in the cases, when the loads (F, M) are applied only at the nodes. When there is distributed load, $v(x)$ is a function of forth order and more accurate solution for this type of beams may be obtained by increasing the number of elements in the model.

3.3.3. 2D –beam

The element is subjected to transverse and axial loads. The stiffness matrix of the element may be obtained superposing the stiffness matrixes (3.6) and (3.78)

$$[k]=\begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ & 12EJ/L^3 & 6EJ/L^3 & 0 & -12EJ/L^3 & 6EJ/L^2 \\ & & 4EJ/L & 0 & -6EJ/L^2 & 2EJ/L \\ & & & AE/L & 0 & 0 \\ & & & & 12/L^3 & -6EJ/L^2 \\ symmetric & & & & & 4EJ/L \end{bmatrix} \qquad (3.82)$$

The matrixes of the nodal displacements of the element in the local and the global coordinate systems are

$$\{\bar{d}\}^T=\lfloor \bar{u}_1\ \bar{v}_1\ \bar{\theta}_1\ \bar{u}_2\ \bar{v}_2\ \bar{\theta}_2 \rfloor \text{ and } \{d\}^T=\lfloor u_1\ v_1\ \theta_1\ u_2\ v_2\ \theta_2 \rfloor \qquad (3.83)$$

The matrix (3.82) is for the local coordinate system. The transformation of the displacements from the local to the global coordinate system is obtained from the relations

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}=\begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \text{ or } \{\bar{d}\}=[T]\{d\} \qquad (3.84)$$

, where $[T]$ is the transformation matrix. It is easily determined that $\{\bar{r}\}=[T]\{r\}$ and since $[T]$ is orthogonal, the stiffness matrix in the global coordinate system may be obtained via the equation

$$[k]=[T]^T[\bar{k}][T] \qquad (3.85)$$

3.3.4. 3D – beam element

In Fig. 11 is shown a finite 3D element in global *xyz* and local $\bar{x}\,\bar{y}\,\bar{z}$ coordinate systems. The nodal degrees of freedom in the local and the global coordinate systems are

$$\{\bar{d}\}^T=\lfloor \bar{u}_i\ \ \bar{v}_i\ \ \bar{w}_i\ \ \theta_{\bar{x},i}\ \ \theta_{\bar{y},i}\ \ \theta_{\bar{z},i} \rfloor \text{ и } \{d\}^T=\lfloor u_i v_i w_i \theta_{x,i}\theta_{y,i}\theta_{z,i} \rfloor,\ i=1,\,2 \qquad (3.86)$$

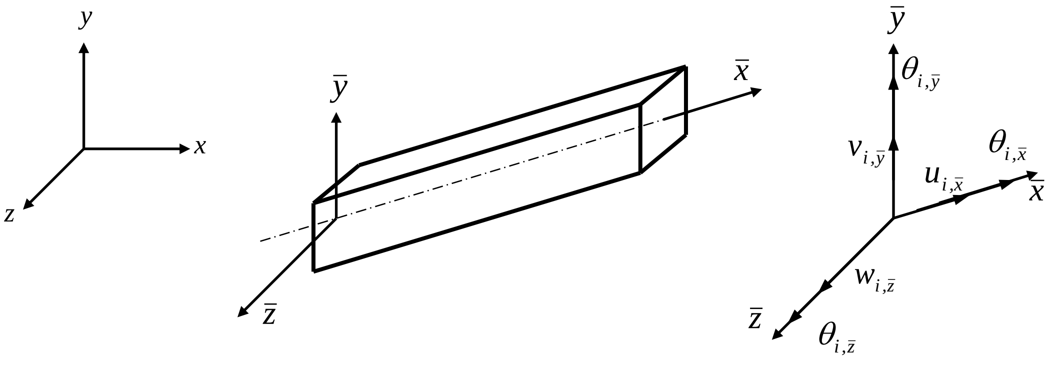


Fig. 3.12

The transformation of the parameters from the local to the global coordinate system is obtained by the relations

$$\begin{Bmatrix} \bar{u}_i \\ \bar{v}_i \\ \bar{w}_i \\ \theta_{\bar{x},i} \\ \theta_{\bar{y},i} \\ \theta_{\bar{z},i} \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 & 0 & 0 & 0 \\ l_2 & m_2 & n_2 & 0 & 0 & 0 \\ l_3 & m_3 & n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 & m_1 & n_1 \\ 0 & 0 & 0 & l_2 & m_2 & n_2 \\ 0 & 0 & 0 & l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \theta_{x,i} \\ \theta_{y,i} \\ \theta_{z,i} \end{Bmatrix} \quad \text{or} \quad \{\bar{d}\} = [T]\{d\} \tag{3.87}$$

i=1 ,2.

, where $[T]$ is the transformation matrix. The element has three forces and three moments per node.

The stiffness matrix in the local coordinate system may be obtained using the method of superposition and adding the matrixes for bending and tension (compression) in *xy* plane (3.82), bending in *xz* plane and torsion.

If the right hand rule is used for the *xz* plane we have:

$$\theta_{\bar{y},1} = -\frac{d\bar{w}(0)}{dx} \text{ и } \theta_{\bar{y},2} = -\frac{d\bar{w}(L)}{dx} \tag{3.88}$$

Then the stiffness matrix for bending in this plane is

$$[\bar{k}] = \frac{EJ_2}{L^3} \begin{bmatrix} 12 & \text{-}6L & \text{-}12 & \text{-}6L \\ & 4L^2 & 6L & 2L^2 \\ & & 12 & 6L \\ \text{symmetric} & & & 4L^2 \end{bmatrix} \tag{3.89}$$

For the stiffness matrix in the case of pure torsion it may be written (see example 3.6)

$$[\bar{k}] = \frac{GJ_c}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.90}$$

Now for the stiffness matrix of a 3D element subjected to bending, tension (compression) and torsion it may be written

$$[\bar{k}] = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{12EJ_1}{L^3} & 0 & 0 & 0 & \frac{6EJ_1}{L^2} & 0 & \frac{-12EJ_1}{L^3} & 0 & 0 & 0 & \frac{6EJ_1}{L^2} \\
 & & \frac{12EJ_2}{L^3} & 0 & -\frac{6EJ_2}{L^2} & 0 & 0 & 0 & \frac{-12EJ_1}{L^3} & 0 & 0 & \frac{-6EJ_1}{L^2} \\
 & & & \frac{GJ_c}{L} & 0 & 0 & 0 & 0 & \frac{-GJ_c}{L} & 0 & 0 & \\
 & & & & \frac{4EJ_2}{L} & 0 & 0 & 0 & \frac{6EJ_2}{L^2} & 0 & \frac{12EJ_2}{L^3} & 0 \\
 & & & & & \frac{4EJ_1}{L} & 0 & \frac{-6EJ_1}{L^2} & 0 & 0 & 0 & \frac{12EJ_1}{L} \\
 & & & & & & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{12EJ_1}{L^3} & 0 & 0 & 0 & \frac{-6EJ_1}{L^2} \\
 & & & & & & & & \frac{12EJ_2}{L^3} & 0 & \frac{6EJ_2}{L^2} & 0 \\
 & & & & & & & & & \frac{GJ_c}{L} & 0 & 0 \\
 & & symmetric & & & & & & & & \frac{4EJ_2}{L} & 0 \\
 & & & & & & & & & & & \frac{4EJ_1}{L}
\end{bmatrix} \qquad (3.91)$$

, where J_1 and J_2 are the second moments of inertia for xy and xz planes, respectively. The transformation of the local stiffness matrix is obtained using the equation

$$[k] = [T]^T [\bar{k}][T] \qquad (3.92)$$

The normal stresses from the internal force N which are $\sigma_x = \frac{N}{A}$, from M_y $\sigma_x = \frac{M_y}{J_y} z$ and from M_z $\sigma_x = \frac{M_z}{J_z} y$ are added algebraically. The torsion moment M_x about x axis produces shearing stresses $\tau_x = \frac{M_x}{J_c} c$, where J_c and c are calculated according to the shape of the cross section.

3.3.5. FEA for "Timoshenko" beams

From the equation for the strain energy (3.35) it is seen that the independent functions of the deformation are two, for plane *xy* – $v(x)$ and $\theta(x)$. For two nodes element (Fig. 3.8,*a*) in the case of linear functions $v(x)$ and $\theta_y(x)$, it may be written

$$\begin{Bmatrix} v \\ \theta \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \qquad (3.93)$$

, where $N_1 = 1 - \frac{x}{L}$ и $N_2 = \frac{x}{L}$.

For isoparametric element $x=\sum_{i=1}^{2} N_i x_i$. For the curvature and the angle of rotation of the cross section, it may be written

$$k=\frac{d\theta}{dx}=\sum_{i=1}^{2}\frac{\partial N_i}{\partial x}\theta_i \qquad (3.94)$$

and

$$\phi=\sum_{i=1}^{2} N_i\theta_i-\sum_{i=1}^{2}\frac{\partial N_i}{\partial x}v_i \qquad (3.95)$$

The strains are determined from the equation

$$\{\varepsilon\}^T=\lfloor k\ \ \phi\rfloor=[B]\{d\} \qquad (3.96)$$

, where

$$[B]=\begin{bmatrix} 0 & \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} \\ \frac{-\partial N_1}{\partial x} & N_1 & \frac{-\partial N_2}{\partial x} & N_2 \end{bmatrix} \qquad (3.97)$$

Taking in account that the matrix of the elastic constants for rectangular cross section is

$$[E]=\frac{Ebh}{12}\begin{bmatrix} h^2 & 0 \\ 0 & \frac{6\alpha}{1+\mu} \end{bmatrix}=\frac{Ebh}{12}\begin{bmatrix} h^2 & 0 \\ 0 & \alpha_1 \end{bmatrix},\ \ \alpha_1=\frac{6\alpha}{1+\mu} \qquad (3.98)$$

the stiffness matrix for the element may be obtained in the local coordinate system as follows

$$[\bar{k}]=\frac{EA}{12L^2}\int_0^L\begin{bmatrix} 6\alpha_1 & 6\alpha_1(L-x) & -6\alpha_1 & 6\alpha_1 x \\ & h^2+6\alpha_1(L-x)^2 & -6\alpha_1(L-x) & -h^2+6\alpha_1 x(L-x) \\ & & 6\alpha_1 & -6\alpha_1 x \\ сим. & & & h^2+6\alpha_1 x^2 \end{bmatrix} \qquad (3.99)$$

If the bending and the shear are separated, we have

$$[\bar{k}]=\frac{EA}{12L^2}\left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h^2 & 0 & -h^2 \\ 0 & 0 & 0 & 0 \\ 0 & -h^2 & 0 & h^2 \end{bmatrix}+6\alpha_1\begin{bmatrix} 1 & (L-x) & -1 & x \\ (L-x) & (L-x)^2 & -(L-x) & x(L-x) \\ -1 & -(L-x) & 1 & -x \\ x & x(L-x) & -x & x^2 \end{bmatrix}\right]dx \qquad (3.100)$$

The integrals can be solved analytically and for rectangular cross section with thickness b = 1 it may be obtained

$$[\bar{k}_{ог}]\frac{Eh^3}{12L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},\ [\bar{k}_{ср}]=\begin{bmatrix} 1 & \frac{L}{2} & -1 & \frac{L}{2} \\ \frac{L}{2} & \frac{L^2}{3} & -\frac{L}{2} & \frac{L^2}{6} \\ -1 & -\frac{L}{2} & 1 & -\frac{L}{2} \\ \frac{L}{2} & \frac{L^2}{6} & -\frac{L}{2} & \frac{L^2}{3} \end{bmatrix} \qquad (3.101)$$

3.3.6. Isoparametric 3-node element for “Timoshenko” beam

` The element is shown in Fig. 1.14.

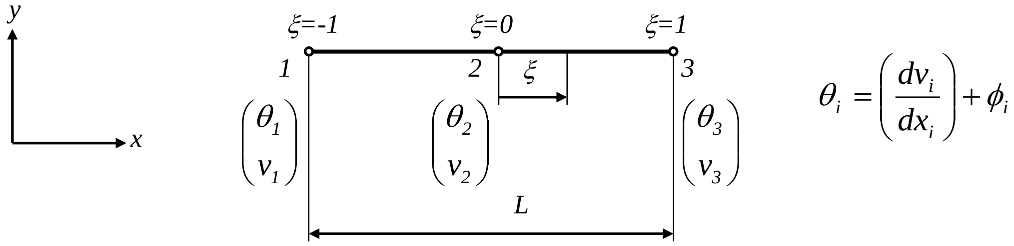


Fig.3.13

The matrix for the nodal displacements is

$$\{d\}^T = \lfloor v_1\ \theta_1\ v_2\ \theta_2\ v_3\ \theta_3 \rfloor \qquad (3.102)$$

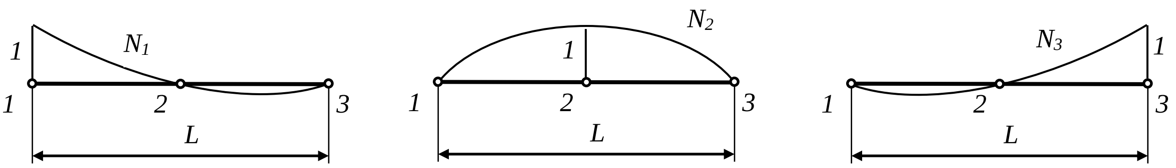
, where θ_i $(\text{i} = 1,2,3)$ for thick beam is determined by

$$\theta_i = \left(\frac{dv}{dx}\right)_i + \phi_i \qquad (3.103)$$

The shape functions are

$$N_1 = -\frac{1}{2}\xi(1-\xi),\ N_2 = (1-\xi)(1+\xi),\ N_3 = \frac{1}{2}\xi(1+\xi) \qquad (3.104)$$

The graphs of the functions are shown in Fig. 3.14.



Фиг. 3.14

The approximating functions are

$$v(x) = N_1(\xi)v_1 + N_2(\xi)v_2 + N_3(\xi)v_3 = \sum_{i=1}^{3} N_i v_i$$

$$\theta(\xi) = N_1(\xi)\theta_1 + N_2(\xi)\theta_2 + N_3(\xi)\theta_3 = \sum_{i=1}^{3} N_i \theta_i \qquad (3.105)$$

$$x(\xi) = N_1(\xi)x_1 + N_2(\xi)x_2 + N_3(\xi)x_3 = \sum_{i=1}^{3} N_i x_i$$

The Jacobian for the element is

$$J = \frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 = \frac{L}{2} \qquad (3.106)$$

The strains are determined from

$$\begin{Bmatrix} \frac{d\theta}{dx} \\ \phi = -\frac{dv}{dx} + \theta \end{Bmatrix} = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} \\ -\frac{\partial N_1}{\partial x} & N_1 & -\frac{\partial N_2}{\partial x} & N_2 & -\frac{\partial N_3}{\partial x} & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = [B]\{d\}, \quad (3.107)$$

,where $d\theta/dx$ is the pseudo-curvature of the beam.

In (3.107) the derivatives are determined from

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N_i}{\partial \xi} J^{-1} \quad (3.108)$$

The relations between the internal forces and the stresses are

$$\begin{Bmatrix} M \\ Q \end{Bmatrix} = \begin{bmatrix} EJ & 0 \\ 0 & \alpha \mathrm{GA} \end{bmatrix} \begin{Bmatrix} \frac{d\theta}{dx} \\ \phi \end{Bmatrix} \quad (3.109)$$

The matrix for the stiffness of the element $[k]$ is 6x6, a typical sub-matrix inside it is determined as

$$[k_{ij}] = \int_{-1}^{1} [B_i]^T [E][B_j] J | d\xi \quad (3.110)$$

or

$$[k_{ij}] = \int \left[\begin{bmatrix} 0 & -\frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial x} & N_i \end{bmatrix} \begin{bmatrix} EJ & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial N_j}{\partial x} \\ -\frac{\partial N_j}{\partial x} & 0 \end{bmatrix} \right] |J| d\xi \quad (3.111)$$

3.4. Thermal stresses

The thermal stresses in the bar and beam of a structure are analyzed by FEA using the general procedure for calculation of thermal stresses in a arbitrary FEA model. This procedure is executed automatically from the software code and includes the following steps, which the user has to activate:

- 1. To take away the element nodes' all degrees of freedom and to calculate the nodal loads from the temperature variation;
- 2. Assemble the elements and the element's loads, calculated in step 1. The mechanical loads are superposed with the thermal ones. As a result, a new model comes out, still unreformed, which nodal loads are a result form the temperature deviations;
- 3. Calculate the nodal displacements, then the deformations in the elements from these nodal displacements, and finally calculate the stresses by the deformations. The calculations are like these made for calculations of the stresses from external loading;
- 4. The stresses from step 3 and the initial stresses from step 1 are summed together, caused by the temperature variations.

The application of the general procedure for determining the thermal stresses in a truss node could be seen in example 3.3.

3.5. Examples
Truss

Example 3.1 For the truss shown in the Fig. 3.15 determine $u\left(\frac{L}{3}\right)$. Given: q, L, $AE = const$

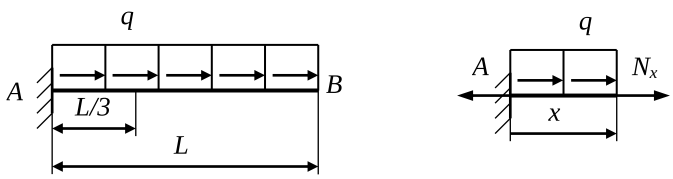


fig. 3.15

SOLUTION

The truss may be considered as one element with two nodes, where the approximation of $u(x)$ is done with linear function.

According to (3.6) it may be written

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} R_A + \frac{qL}{2} \\ \frac{qL}{2} \end{Bmatrix}$$

where from we obtain $u_2 = \frac{qL^2}{2AE}$. R_A is the support reaction at A. According to (2.28), since $u_1 = 0$, $u(x) = N_2 u_2 = \frac{x}{L}\frac{qL^2}{2AE}$, so we obtain the value we are looking for and the unknown value of the displacement $u\left(\frac{L}{3}\right) = \frac{qL^2}{6AE}$.

The exact solution is $\Delta L(x) = \int_0^x \frac{N(x)}{AE}dx = \frac{1}{AE}\int_0^x q(L-x)dx = \frac{1}{AE}q\left(Lx - \frac{x^2}{2}\right)$.

For $x = \frac{L}{3}$ we obtain $u\left(\frac{L}{3}\right) = \frac{5}{18}\frac{qL^2}{AE}$.

The error for the solution with one two nods element is $\Delta = 40\%$.

The accuracy could be increased by increasing the number of the elements or the number of the nodes.

A) Solution with increased number of elements.

The solution is done by a model of two two-nodes elements (fig. 3.16). Both of the nodes are lying on one line and the directions of the axis of the local coordinate system coincide with these

fig. 3.16

According to (3.6) for the both elements we can write

$$\frac{AE}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 \end{Bmatrix}; \qquad \frac{AE}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = \begin{Bmatrix} F_1^2 \\ F_2^2 \end{Bmatrix}$$

as

$$\begin{matrix} u_1^1 = u_1, \; u_2^1 = u_2 \\ u_1^2 = u_2, \; u_2^2 = u_3 \end{matrix}.$$

Considering the fact that, $F_1^1 = F_1 = R_A$, $F_2^1 + F_1^2 = F_2 = 0$ *and* $F_2^2 = F_3 = 0$ after the assembling of the system we get:

$$\frac{AE}{L/2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} R_A + \dfrac{qL}{4} \\ \dfrac{ql}{2} \\ \dfrac{qL}{4} \end{Bmatrix}.$$

Because $u_1 = 0$ the system is reduced to

$$\frac{AE}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{qL}{2} \\ \dfrac{qL}{4} \end{Bmatrix}$$, so we get $u_2 = \dfrac{3}{8}\dfrac{qL^2}{AE}$, $u_3 = \dfrac{1}{2}\dfrac{qL^2}{AE}$.

The $\left(\dfrac{L}{3}\right) = \dfrac{qL^2}{4AE}$. The error is $\Delta = 10\%$.

The error is decreased significantly, if we use a model with 3 elements.

B) Solution with increasing the number of the nodes.

We use one finite element with 3 nodes (fig. 3.17).

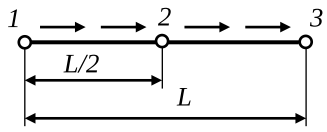


fig. 3.17

For this element the approximation of the displacements is done with function of second order:

$$u(x) = a_1 + a_2 x + a_3 x^2$$

In order to determine the polynomial coefficients we use the system of equations

$$u_1 = a_1$$

$$u_2 = a_1 = a_1 + a_2\frac{L}{2} + a_3\frac{L^2}{4}$$

$$u_3 = a_1 + a_2 L + a_3 L$$

After that,the approximating function could be presented that way:

$$u(x) = N_1 u_1 + N_2 u_2 + N_3 u_3,$$

where $N_1 = \left(\dfrac{2x}{L} - 1\right)\left(\dfrac{x}{L} - 1\right)$, $N_2 = \dfrac{4x}{L}\left(1 - \dfrac{x}{L}\right)$ и $N_3 = \dfrac{x}{L}\left(2\dfrac{x}{L} - 1\right)$

For the stiffness matrix of the element, considering that:

$$[B] = [D][N] = \left\lfloor \frac{4x}{L^2} - \frac{3}{L} \quad \frac{4}{L} - \frac{8x}{L^2} \quad \frac{4x}{L^2} - \frac{2x}{L} \right\rfloor$$

we obtain

$$[k]=\frac{AE}{L}\begin{bmatrix}\frac{7}{3} & -\frac{8}{3} & \frac{1}{3}\\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3}\\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3}\end{bmatrix}$$

For $\{r\}$ we get $\{r\}=q\int_0^L [N]^T dx = q\int_0^L \begin{bmatrix}\left(\frac{2x}{L}-1\right)\left(\frac{x}{L}-1\right)\\ \frac{4x}{L}\left(1-\frac{x}{L}\right)\\ \frac{x}{L}\left(2\frac{x}{L}-1\right)\end{bmatrix} dx$ or $\{r\}=\begin{Bmatrix}R_A+\frac{1}{6}qL\\ \frac{2}{3}qL\\ \frac{1}{6}qL\end{Bmatrix}$

The system of equations is

$$\frac{AE}{L}\begin{bmatrix}\frac{7}{3} & -\frac{8}{3} & \frac{1}{3}\\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3}\\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}R_A+\frac{1}{6}qL\\ \frac{2}{3}qL\\ \frac{1}{6}qL\end{Bmatrix}$$

Because $u_1=0$ the system is reduced to $\frac{AE}{L}=\begin{bmatrix}\frac{16}{3} & -\frac{8}{3}\\ -\frac{8}{3} & \frac{7}{3}\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}\frac{7}{3}qL\\ \frac{1}{6}qL\end{Bmatrix}$, se we can obtain $u_2=\frac{3}{8}\frac{qL^2}{AE}$ and $u_3=\frac{1}{2}\frac{qL^2}{AE}$.

For the displacements function we can write

$$u(x)=\frac{4x}{L}\left(1-\frac{x}{L}\right)\frac{3}{8}\frac{qL^2}{AE}+\frac{x}{L}\left(\frac{2x}{L}-1\right)\frac{1}{2}\frac{qL^2}{AE}$$

and the searched displacement is given by $u\left(\frac{L}{3}\right)=\frac{5}{18}\frac{qL^2}{AE}$. The solution in the FESA method done with one 3 nodes finite element coincides with the exact one.

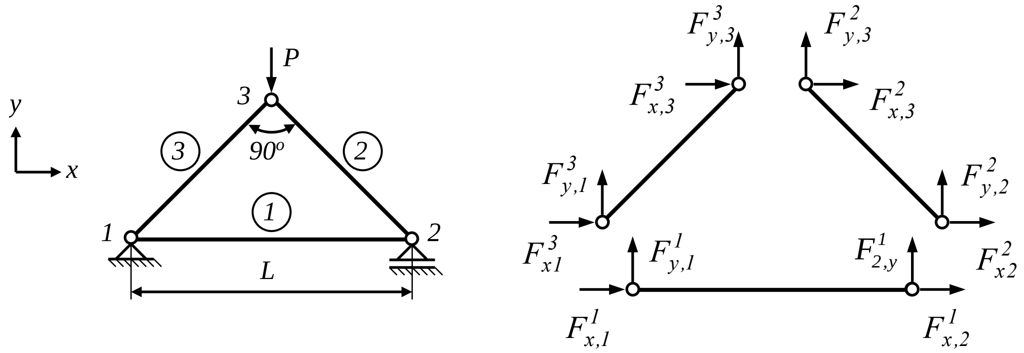
The deformations are determined by (1.12). After replacing of $[B]$ and $\{u\}$ we get $\varepsilon(x)=\frac{qL}{AE}-\frac{qx}{AE}$. The function gives the exact solution for the deformations along the truss.

The stresses are determined by the Hook's law

$$\sigma(x)=E\left(\frac{qL}{AE}-\frac{qL}{AE}\right)$$

The function gives the exact solution for the stresses along the truss.

Example 3.2 For the shown in fig. 3.18, a truss structure, determine the forces in the trusses and the displacements of nodes 2 and 3. Given is $A_1E_1 = A_2E_2 = A_3E_3 = AE, L, P$.



a) b)

Fig. 3.18

The structure is separated into 3 elements 1,2 and 3 and has 3 nodes 1, 2 and 3.
Using (3.16) we can obtain the transforming matrices of the 3 elements:

$$[T_1] = \begin{bmatrix} 1 0 0 0 \\ 0 0 1 0 \end{bmatrix}, [T_2] = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, [T_3] = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 1 0 0 \\ 0 0 1 1 \end{bmatrix}$$

The from (3.21) the stiffness matrices are defined

$$[k_1] = \frac{A_1E_1}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [k_2] = \frac{A_2E_2}{L\frac{\sqrt{2}}{2}}\left(\frac{\sqrt{2}}{2}\right)^2\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$[k_3] = \frac{A_3E_3}{L\frac{\sqrt{2}}{2}}\left(\frac{\sqrt{2}}{2}\right)^2\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

in the global coordinate system.

For convenience when assembling the system, these matrices could be expanded from 4x4 to 6x6, as we put zero rows and columns with numbers, according to the number of the missing rows in the column of the displacements. Like that $k_1\{d_1\} = \{F_1\}$ is transformed in this way

$$\frac{A_1E_1}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{x,1}^1 \\ F_{y,1}^1 \\ F_{x,2}^1 \\ F_{y,2}^1 \end{Bmatrix}, \quad \frac{A_1E_1}{L}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{x,1}^1 \\ F_{y,1}^1 \\ F_{x,2}^1 \\ F_{y,2}^1 \\ 0 \\ 0 \end{Bmatrix} \qquad (3.112)$$

Analogically we get the matrix forms for the other trusses too

$$\frac{A_2E_2}{L\frac{\sqrt{2}}{2}}\left(\frac{\sqrt{2}}{2}\right)^2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ F_{x,2}^2 \\ F_{y,2}^2 \\ F_{x,3}^2 \\ F_{y,3}^2 \end{Bmatrix} \tag{3.113}$$

$$\frac{A_3E_3}{L\frac{\sqrt{2}}{2}}\left(\frac{\sqrt{2}}{2}\right)^2\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix}=\begin{Bmatrix} F_{x,1}^3 \\ F_{y,1}^3 \\ 0 \\ 0 \\ F_{x,3}^3 \\ F_{y,3}^3 \end{Bmatrix}. \tag{3.114}$$

The assembling the system of equations is done on the basis of the conditions for the equilibrium in the nodes (fig. 3.18, b).

$$\begin{matrix} F_{x,1}^1+F_{x,1}^3=F_{x,1} & F_{x,2}^1+F_{x,2}^2=F_{x,2} & F_{x,3}^2+F_{x,3}^3=F_{x,3} \\ F_{y,1}^1+F_{y,1}^3=F_{y,1} & F_{y,2}^1+F_{y,2}^2=F_{y,2} & F_{y,3}^2+F_{y,3}^3=F_{y,3} \end{matrix} \tag{3.115}$$

Moreover

$$F_{x,1}=0,\ F_{y,1}=0,\ F_{x,2}=0,\ F_{y,2}=0,\ F_{x,3}=0,\ F_{y,3}=-P \tag{3.116}$$

On the basis of (3.112) $\div$ (3.116) we get

$$\frac{AE}{L\sqrt{2}}\begin{bmatrix} 1+\sqrt{2} & 1 & -\sqrt{2} & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ -\sqrt{2} & 0 & 1+\sqrt{2} & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 2 & 0 \\ -1 & -1 & 1 & -1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}=\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{Bmatrix}. \tag{3.117}$$

Because $u_1=v_1=v_2=0$, then the system (1.69) is reduced with removing rows and columns in the matrices, with the number corresponding to the number of the rows and columns with zero displacements. Thus we obtain:

$$\frac{AE}{L\sqrt{2}}\begin{bmatrix} 1+\sqrt{2} & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}. \tag{3.118}$$

From the solution of (3.118) we get

$$u_2 = \frac{PL}{2AE}, \quad u_3 = \frac{PL}{4AE}, \quad v_3 = -\frac{PL}{4AE}\left(1+2\sqrt{2}\right)$$

The truss forces are derived according to the relations (3.112) ÷ (3.114). For example the truss force in truss 1 ,according to (3.112) is $F_{1,x}^{1} = \frac{AE}{L}u_{2,x} = -\frac{P}{2}$ and so on.

Example 3.3

Determine the stresses in the truss from fig. 3.19, given: *A*, *E*, *L*, *P*. After the mounting of the rod (truss), it's temperature increases uniformly with ΔT . In the right hand end, the support bends as a linear spring wit stiffness $k = \frac{AE}{L}$.

The truss is shown in fig. 3.19, a and is divided into 2 elements. The result from the temperature variation are the forces $F_T = \alpha AE\Delta T$, shown on the both elements (fig. 3.19, a and fig. 3.19, b). The spring in the right end could be replaced with force $P_S = \frac{AE}{L}u_3$ (Fg. 3.19, c).

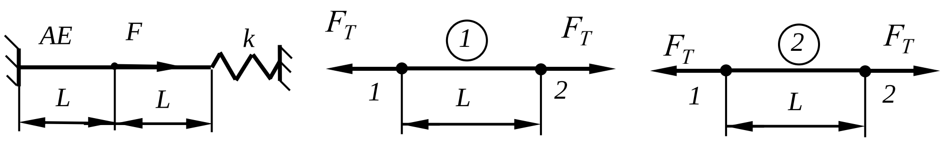


fig. 3.19

First off, according to the rules, in the joints of the structure we define the loads in the nodes of the elements, caused by the temperature increase

$$F_T = \frac{2}{3}\alpha\Delta TAE$$

From the relations

$$\begin{aligned} &F_{11} = R_1 - F_T = r_1 \\ &F_{21} + F_{12} = P + F_T - F_T = r_2 \\ &F_{22} = F_T - P_s = F_T - \frac{AE}{L}u_3 = r_3 \end{aligned} \tag{3.119}$$

we could assemble the elements and to write the global system of equations

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R_1 - F_T \\ P + F_T - F_T \\ F_T - \frac{AE}{L}u_3 \end{Bmatrix} \tag{3.120}$$

where R_1 is the reaction in node *1*, that is considered unknown. The summing of the forces is done with positive direction on the right.

Since the support on the left is still $u_1 = 0$ and then the system (3.120) is reduced to

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_T - \frac{AE}{L}u_3 \end{Bmatrix}. \tag{3.121}$$

The solution of (3.121) is

$$u_2 = \frac{2}{3}\frac{PL}{AE} + \frac{1}{3}\frac{L}{AE}F_T \text{ и } u_3 = \frac{1}{3}\frac{FL}{AE} + \frac{2}{3}\frac{L}{AE}F_T$$

The stresses in the elements are defined as

$$\sigma_{i,x} = E\varepsilon_{i,x} + \sigma_{i,xo}, i = 1, 2$$

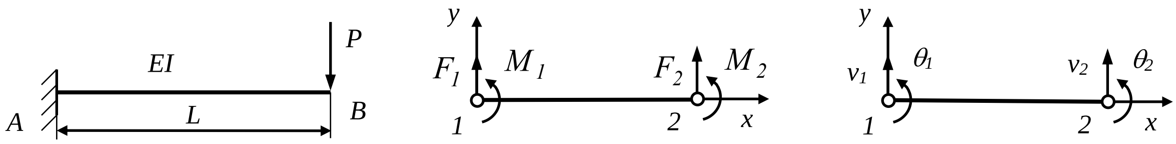
where $\sigma_{1,xo} = -\frac{F_T}{A}$ is the initial stress, caused by the temperature increase. So for the stresses we obtain

$$\sigma_{x,1} = E\frac{u_2 - u_1}{L} + (-\frac{F_T}{A}). = \frac{2}{3}\frac{F}{A} - \frac{2}{3}\frac{F_T}{A} = \frac{2}{3}\frac{F}{A} - \frac{4}{9}\alpha E\Delta T$$
$$\sigma_{x,2} = E\frac{u_3 - u_2}{L} + (-\frac{F_T}{A}). = -\frac{1}{3}\frac{F}{A} - \frac{2}{3}\frac{F_T}{A} = -\frac{1}{3}\frac{F}{A} - \frac{4}{9}\alpha E\Delta T$$

Beam Structures

Example 3.4

Determine the displacement v_B and the rotation θ_B in a cross section *B* from the shown on fig. 3.20, a beam. Given is: EJ, L, P.



a) b c

fig.20

SOLUTION

Since $EJ = const$ the whole beam could be considered as one element. In that case in the matrix of the nodal loads, there will be only one nodal force in node 2, because $F_1^1 = F_1,\ M_1^1 = M_1$ и $F_2^1 = P,\ M_2^1 = 0$ (Fig 3.20, *b*), as F_1 and M_1 are the support reactions. Then according to relations (3.63) and (3.64) we have

$$EJ\begin{bmatrix} \frac{4}{L} & \frac{2}{L} & \frac{6}{L^2} & -\frac{6}{L^2} \\ \frac{2}{L} & \frac{4}{L} & \frac{6}{L^2} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{6}{L^2} & \frac{12}{L^3} & -\frac{12}{L^3} \\ -\frac{6}{L^2} & -\frac{6}{L^2} & -\frac{12}{L^3} & \frac{12}{L^3} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} M_1 \\ M_2 \\ F_1 \\ F_2 \end{Bmatrix}$$

Since θ_1 and v_1 are equal to *0*, then the upper system of equations is reduced by removing of rows and columns of the stiffness matrix with the numbers corresponding to the numbers of the row with the displacement equal to 0. Moreover $M_2 = 0$ and $F_2 = -P$. Thus we get the system of equations in matrix form

$$EJ\begin{bmatrix} \frac{4}{L} & -\frac{6}{L^2} \\ -\frac{6}{L^2} & \frac{12}{L^2} \end{bmatrix}\begin{Bmatrix} \theta_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\text{P} \end{Bmatrix}$$

where we can define $\theta_2 = \theta_B = -\dfrac{PL^2}{2EJ}$ and $v_2 = v_B = -\dfrac{PL^3}{3EJ}$.

The obtained results show, that the solution coincides with the exact solution (look at the tables for strength of materials). The deformations are defined from (3.71).

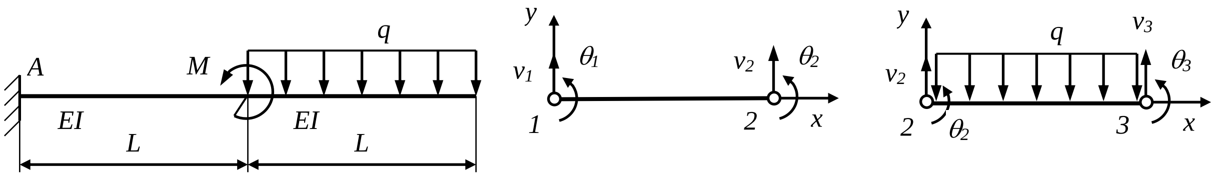
It is easy to see that, the function $\varepsilon(x) = -(\dfrac{qL}{EJ} - \dfrac{qx}{EJ})z$ gives the solution of the technical theory of elasticity.

For the stresses we get

$$\sigma(x) = E.\varepsilon(x) = -E\left(\frac{qL}{EJ} - \frac{qx}{EJ}\right)z$$

Example 3.5

For the shown on Fig. 3. 21, a beam, determine the droop and the rotationof the cross sections *B* and *C*. Given is: $EJ = const, q, L, M$, q.



a) b) c)

fig. 3.21

SOLUTION

The beam is divided to 2 finite elements (fig. 3.21, b and c). For the elements *1* and *2* we can write

$$EJ\begin{bmatrix} \frac{4}{L} & \frac{2}{L} & \frac{6}{L^2} & -\frac{6}{L^2} \\ \frac{2}{L} & \frac{4}{L} & \frac{6}{L^2} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{6}{L^2} & \frac{12}{L^3} & -\frac{12}{L^3} \\ -\frac{6}{L^2} & -\frac{6}{L^2} & -\frac{12}{L^3} & \frac{12}{L^3} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} M_1^1 \\ M_2^1 \\ F_1^1 \\ F_2^1 \end{Bmatrix}, \quad EJ\begin{bmatrix} \frac{4}{L} & \frac{2}{L} & \frac{6}{L^2} & -\frac{6}{L^2} \\ \frac{2}{L} & \frac{4}{L} & \frac{6}{L^2} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{6}{L^2} & \frac{12}{L^3} & -\frac{12}{L^3} \\ -\frac{6}{L^2} & -\frac{6}{L^2} & -\frac{12}{L^3} & \frac{12}{L^3} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} M_1^2 \\ M_2^2 \\ F_1^2 \\ F_2^2 \end{Bmatrix} \begin{Bmatrix} \frac{qL^2}{12} \\ -\frac{qL^2}{12} \\ \frac{qL}{2} \\ \frac{qL}{2} \end{Bmatrix}.$$

.

From the equilibrium conditions for node 2, we can write: $F_1^1 = F_1$, $M_1^1 = M_1$, as M_1 and F_1 are support reactions, $F_2^1 + F_1^2 = F_2 = -\dfrac{qL}{2}$ and $M_2^1 + M_1^2 = M - \dfrac{qL^2}{2}$, as $F_2^1, M_2^1, F_1^2, M_1^2$ are the nodal forces and moments in nodes *2* and *1*, respectively of elements 1 and 2 and

$F_1^2 = F_3 = -\dfrac{qL}{2}$, $M_2^2 = M_3 = \dfrac{qL^2}{2}$. On the basis of these equations is done the assembling of the upper system of equations. Thus we have

$$EJ\begin{bmatrix} \frac{4}{L} & \frac{2}{L} & 0 & \frac{6}{L^2} & -\frac{6}{L^2} & 0 \\ \frac{2}{L} & \frac{8}{L} & \frac{2}{L} & \frac{6}{L^2} & 0 & -\frac{6}{L^2} \\ 0 & \frac{2}{L} & \frac{4}{L} & 0 & \frac{6}{L^2} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{6}{L^2} & 0 & \frac{12}{L^3} & -\frac{12}{L^3} & 0 \\ -\frac{6}{L^2} & 0 & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{24}{L^3} & -\frac{12}{L^3} \\ 0 & -\frac{6}{L^2} & -\frac{6}{L^2} & 0 & -\frac{12}{L^3} & \frac{12}{L^3} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} M_1 \\ M - \frac{qL^2}{12} \\ \frac{qL^2}{12} \\ F_1 \\ -\frac{qL}{2} \\ -\frac{qL}{2} \end{Bmatrix}$$

Because $\theta_1 = 0$ and $v_1 = 0$ the system is reduced to

$$EJ\begin{bmatrix} \frac{8}{L} & \frac{2}{L} & 0 & -\frac{6}{L^2} \\ \frac{2}{L} & \frac{4}{L} & \frac{6}{L^2} & -\frac{6}{L^2} \\ 0 & \frac{6}{L^2} & \frac{24}{L^3} & -\frac{12}{L^3} \\ -\frac{6}{L^2} & -\frac{6}{L^2} & -\frac{12}{L^3} & \frac{12}{L^3} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} M - \frac{qL^2}{12} \\ \frac{qL^2}{12} \\ -\frac{qL}{2} \\ -\frac{qL}{2} \end{Bmatrix}$$

where we can obtain

$$v_2 = \frac{ML^2}{2EJ} - \frac{7}{12}\frac{qL^4}{EJ}, \quad \theta_2 = \frac{ML}{EJ} - \frac{qL^3}{EJ}, \quad v_3 = \frac{3}{2}\frac{ML^2}{EJ} - \frac{41}{24}\frac{ql^4}{EJ}, \quad \theta_3 = \frac{ML}{EJ} - \frac{7}{6}\frac{qL^3}{EJ}$$

Example 3.6

Derive the equations, defining the behaviour of the structure from fig. 3.22, *a*, if it is given that *EJ*, *L*, *P*, *q*.

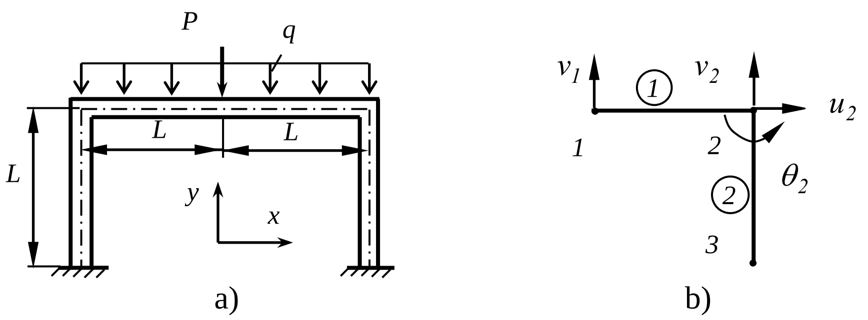


Fig. 3.22

Since the structure has axis of symmetry, we can investigate only the half of it. In the cross section of the axis of symmetry the rotation and the horizontal displacements are zeros. We build a model with 2 elements (Fig. 3.22, *b*).

If we accept the global coordinate system *xy*, for element *1* the stiffness matrix in the local and the global coordinate system is

$$[k]=\begin{bmatrix} c & 0 & 0 & -c & 0 & 0 \\ & 12a & 6aL & 0 & -12a & 6aL \\ & & 4aL^2 & 0 & -6aL & 2aL^2 \\ & & & c & 0 & 0 \\ & & & & 12a & -6aL \\ symmetric & & & & & 4aL^2 \end{bmatrix} \qquad (3.122)$$

where $c=\dfrac{EA}{L}$ and $\grave{a}=\dfrac{EJ}{L^3}$.

The stiffness matrix of element 2 could be derived from (3.85), taking into account, that $\cos\alpha = 0$ and $\sin\alpha=1$. The transforming matrix is

$$[T]=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (3.123)$$

The stiffness matrix of the element 2 in the local coordinate system has the same form as that for element *1*. After doing the operations according to (3.85) we get

$$[k_2]=\begin{bmatrix} 12a & 0 & 6aL & 12a & 0 & 6aL \\ & c & 0 & -c & 0 & 0 \\ & & 4aL^2 & 6aL & 0 & 2aL^2 \\ & & & 12a & 0 & 6aL \\ & & & & c & 0 \\ symmetric & & & & & 4aL^2 \end{bmatrix}. \qquad (3.124)$$

From the conditions for equilibrium of the nodes we can write

$$\begin{gathered} F_{x,1}=F^1_{x,1}=0 \qquad \mathrm{F}_{x,2}=F^1_{x,2}+F^2_{x,1}=0, \quad F_{x,3}=R_{D,x} \\ \mathrm{F}_{y,1}=\mathrm{F}^1_{y,1}=-\frac{P}{2}-\frac{1}{2}qL, \quad \mathrm{F}_{y,2}=F^1_{y,2}+F^2_{y,1}=-\frac{1}{2}qL, \quad F_{y,3}=R_{D,y}, \\ \mathrm{M}_1=M^1_1=0, \quad \mathrm{M}_2=\mathrm{M}^1_2+M^2_1=-\frac{qL^2}{12}, \quad M_3=M_D \end{gathered} \qquad (3.125)$$

where $R_{D,x}$, $R_{D,y}$ and M_D are the support reactions in p. *D*.

The nodal parameters and loads in the global coordinate system are

$$\begin{gathered} \{\Delta\}^T=[u_1\ v_1\ \theta_1\ u_2\ v_2\ \theta_2\ u_3\ v_3\theta_3] \\ \{R\}^T=[F_{x,1}\ F_{y,1}\ M_1\ F_{x,2}\ F_{y,2}\ M_2\ F_{x,3}\ F_{y,3}\ M_3] \end{gathered} \qquad (3.126)$$

Considering $u_1=0,\ \theta_1=0,\ u_3=0,\ v_3=0,\ \theta_3=0$, on the basis of (3.123), (3.124) and (3.125) we obtain

$$\begin{bmatrix} 0 & 12a & -12a & 6aL \\ 0 & c+12a & 0 & 6aL \\ 12a & 0 & c+12a & -6aL \\ 6aL & 6aL & -6aL & 8aL^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{P}{2} - \frac{1}{2}L \\ 0 \\ -\frac{1}{2}qL \\ -\frac{1}{12}qL^2 \end{Bmatrix}. \qquad (3.127)$$

Example 3.6 Derive the stiffness matrix of an element subjected to twisting ,given *G*, J_c, *L*.

On the fig. 3.23, a is shown element of pure twist with one degree of freedom ϑ in a node (fig. 3.23, a). The nodal loads are shown in fig. 3.23, *b*.

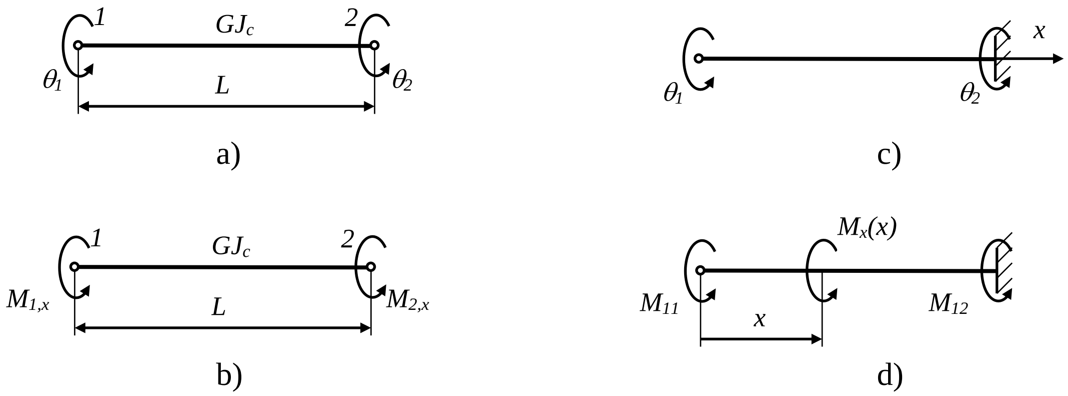


fig. 3.23

If only node 1 gets a nodal rotation ϑ_1 , $M_{1,x}$ (fig. 3.23 c) could be defined as

$$\vartheta_1 = \frac{\partial U}{\partial M_{1,x}} = \int_0^L \frac{M_x(x)}{GJ_c}\frac{\partial M_x}{\partial x}dx = \frac{M_{1,x}}{GJ_c}L \qquad (3.128)$$

where G is the sliding modulus, and J_c is generally speaking inertial characteristics of the cross section.

From (3.128) we can derive $M_{11} = \frac{GJ_c}{L}$.

Form the static equilibrium condition of the element we get $M_{11} = \frac{GJ_c}{L} = -M_{21}$. Analogically, in rotation ϑ_2 of node *2* we get $-M_{12} = \frac{GJ_c}{L} = M_{22}$.

For the resultant of the moments in nodes 1 and 2 we can write

$$M_1 = M_{11} + M_{12,x} = \frac{GJ_c}{L}\vartheta_1 - \frac{GJ_c}{L}\vartheta_2$$

$$M_2 = M_{21} + M_{22} = -\frac{GJ_c}{L}\vartheta_1 + \frac{GJ_c}{L}\vartheta_2$$

(1.52) can be written in matrix form like

$$\frac{GJ_c}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix} = \begin{Bmatrix} M_{1,x} \\ M_{2,x} \end{Bmatrix} = [k]\{d\} = \{r\}. \qquad (3.129)$$