Chapter 2
Basic principles in FEA (Finite Element Analysis)

2.1. Introduction

The Finite Element Analysis is a numerical method for solving a wide variety of problems in the field of mechanics (Geomechanics, mechanics of the rigid deformable body), heat engineering, electrical engineering etc. In he mechanics of rigid deformable body, it is continuation of the Displacements method, initially developed by engineers with physics interests, and then from mathematicians with more abstract methods. As it is done in the method of displacements, using known relations between the forces and the displacements for every single element in the construction, we can make analysis for the behavior of the construction as a whole using the FEA method too. For the first time, the FEA method was applied for analysis of the stress field, and after that for other problems of the mechanics of the continuous mediums. The aim of every research is the determination of the field characteristics, for instance, the distribution of the stresses or the displacements in stress analysis, the values of the stresses and the displacements in the separate points from the field, extreme values of these quantities or their gradient and so on.

A simpler way for presenting the FEA method is, dividing the body (structure) to finite number of parts called finite elements, describing the behavior of each one of them by simple means and then merging them back in points called nodes(fig.2.1.). The interaction between the elements happens in the nodes only, As a result of that process, we get a system of algebraic equations, that stand for static equilibrium equations of the nodes in a stress analysis problem.

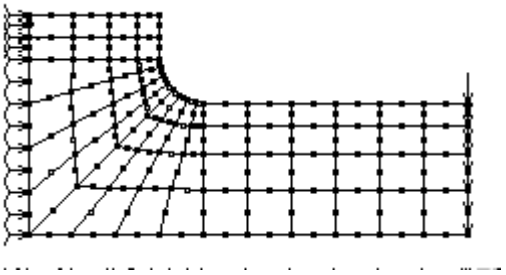


Fig.2.1.

In the area of the finite element ,the field of displacements is approximated by means of the nodal values using approximating functions. The nodal values themselves are defined by minimizing the functional of the full strain energy. As a result, we get a system of algebraic equations with basic unknown nodal displacements. The matrix form of this system is $[K]\{\Delta\}=\{R\}$, where $\{\Delta\}$ is a vector of the unknown value of the field under investigation in the nodes,{R} is a vector of the nodal loads and [K] is a square matrix. If all the loads are transformed on the basis of deformational equivalence in nodal loads, then the vector of the nodal loads will be completely defined. Choosing the approximating characteristics for the finite element field investigation, ad accepting the material properties, the matrix [K] is completely defined to. That matrix is also known as a stiffness matrix of the finite element in the stress analysis, because it interconnects the displacements, respectively the deformations, with the loading. Following the presented formulation above, we may state that exact solution is available only for the nodal values, while it is approximated in the finite element itself. In the so called Variation Principle of FEA method, the main relations about the finite element are a product of the principle of the virtual displacements.

A significant advantage of the FEA method with respect to the analytical methods is that it can be used for solving problems with no restrictions to the shape of the body, arbitrary static boundary conditions (loading) and arbitrary geometric boundary conditions (supports). Of course, this does not mean that the work with the method lacks problems. Constructing the geometric model, assigning proper static and geometric conditions, awareness of the properties of the finite elements constituting the FEA model, could be a result only of a deep study of the basics and the practical applications of the FEA method. Results, taken from a FE analysis need a great amount of experience and knowledge in the field of mechanics in order to make valid conclusions. All commercial software products are thoroughly tested with problems, whose solution could be found different ways. From that point, the responsibility for the results and the conclusions is the hands of the end user.

2.2.Basic Stages in the FEA method.

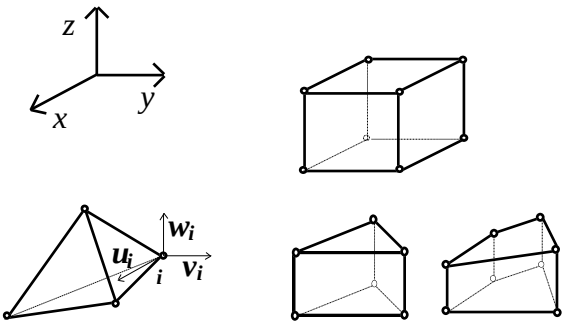
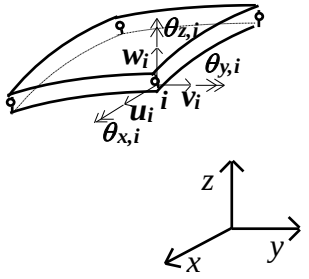
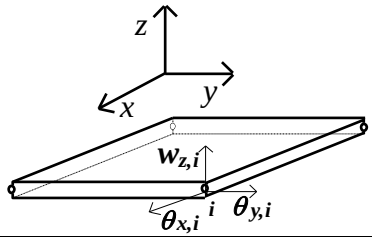
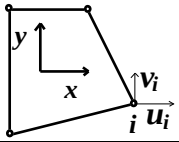
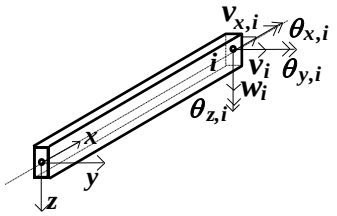
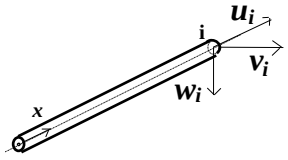
1. Discretisation

The object of investigation is divided into finite number of elements with respect to its shape. This shape is known as division region. The type of the finite elements is predefined with respect to the problem. The software products consists of library of finite elements for solution of different types pf problems. Different types of finite elements with respect to their shape and dimensions could be used in the discretisation of a region in the different sub-regions, following the rule that they must fulfill continuously the whole region of division and the nodes of the neighbor finite elements must coincide.

Regarding the geometry and the hypotheses used for construction of the model, the division region could be a line (axis of a beam or truss), plane (the middle plane of a slab) or 3D shape (f.e. a body). According to this the finite elements could be 1D (linear), 2D (planary), 3D (volumetric).

The 1D finite elements could be straight lines or curves depending on the number of the nodes laid longitudinally over the element. Most used 2D elements are 3-angle or 4-angle ones. The edges of these elements could also be straight lines or curves depending on the number of the nodes on them. 2D finite elements are used for solving the so called 2D problems (2D stress and 2D strain state), and analyzing of bodies with axis-symmetrical shape and loading too. Most often used 3D finite elements are tetrahedron and hexahedron. The walls of these figures could be straight or curved planes, again depending on the number of the nodes on the walls. Different types of finite elements are shown in table 2.1.

Table 2.1

Model,Element	*Application*
3 dimensional model  *tetrahedron* *hexahedron*	*3 dimensional body (shape)*
Shell model  	*thin wall structures* *shells*
Slab model  	*thin wall structures* *slabs*
Planar model  	*body under 2D stress/strain state* *axis-symmetrical bodies*
Beam model  	*beam structures*
Truss model  	*truss structures*

The discretisation is of vital importance in order to reach high accuracy. There are no strict rules, according to which one should create a mesh of finite elements, but usually, the mesh should be denser on the places where we expect to have great gradient of the stresses and strains (fig.2.1.) Increasing the accuracy of the numerical solution could be achieved by increasing the degrees of freedom that could be done in 2 ways: a) increasing the number of the finite elements in the division region (h-version of the FEA method); b)improving the approximation (i.e. increasing the number of the nodes used in the FEs or increasing the degrees of freedom in a node, so called p-version of FEA method) c)redistribution of the existing nodes without changing the number of the elements (r-version).

Increasing the number of the finite elements in the division region we increase the accuracy of the solution but so does the number of the equations to be solved too, that requires more time for calculations. In most cases the greater accuracy could be achieved by increasing the degree of the approximating polynomial (p-version).

2. Approximating the displacements inside the finite element

The displacements of the points on the finite element are determined by the displacements of the nodes. Therefore we use system of functions, called approximating functions, with the help of which the displacements inside the FE are approximated, according to the accepted nodal displacements. The choice of these functions is an important stage in the FEA method. The accuracy of the solution depends entirely on that choice. In fact the discretization is not only an elementary division of the body on finite elements, but a choice of approximating functions. They are usually polynomials. The greater the degree of the polynomial, the better the results in the solution, and the greater number of calculations.

3.Determination of the nodal forces.

For simplification, in FEA method we work only with nodal loads. They are concentrated loads in the nodes of the element. The equivalent nodal loads are defined by the condition for equivalence of the work, done by the nodal loads on the possible nodal displacements and the work of the actual loads on the virtual displacements of the points of application. All the surface and volume loads are transformed into nodal loads, and in order NOT to be transformed the concentric loads, their points of application are chosen to be nodes of the mesh of finite elements. The nodal loads form the so called vector of the nodal loads.

4.Deriving the equations for equilibrium of a finite element.

According to the principle of possible displacements or through minimizing the full strain energy for a given element, one could write equations, relating the nodal displacements and the nodal forces, as the number of the equations is equal to the multiplication of, the number of nodes and the number of degrees of freedom in one node. The result defines the degrees of freedom of the finite element too. The equations are written in matrix form and give the relationship between the nodal forces and the nodal displacements using the so called stiffness matrix of the finite element. The stiffness matrix of the FE is completely defined with the choice of approximation function for the displacements of the FE and the material properties for it. The nodal displacements remain unknown.

5.Assembly the system

The derived equations for the separate FEs can not be solved for themselves in order to determine the nodal displacements. Using the conditions for equilibrium of the nodes it needs to be done the so called Assembling of the system of equations. The result in matrix form consists of the so called Global stiffness matrix and it relates the nodal displacements with the nodal loads over the whole structure.

6.Determination of the nodal displacements

The nodal displacements are determined after solving the assembled system of equations.

7.Determination of the stresses and the strains in a finite element

The displacements inside the finite element are determined by the already defined nodal displacements using the approximating functions. The stresses and the strains in the FE are determined using the relations between the strains and the displacements (equations of Cauchi) and the relations between the strains and the stresses (Hook's law).

2.3. Principle formulation of the FEA method

The FEA method could be defined in different ways. Looking back in the history, it is a result of methods, investigating the equilibrium of multiple elements, used to determine the relations between the forces and the displacements. The FEA method is a direct application and summary of the direct method of the displacements in truss structures for solving 2D and 3D problems in the theory if elasticity. The energy methods that are easily determined to be more effective for statically undetermined structures, are now dominating in the conceptions for the FEA method. The basic dependencies for the finite element and the final system of equations are generated using the principle of the virtual displacements or the principle of stationarity of the full strain energy, that for the so called conservative systems is turned into a principle for the minimum of the full strain energy.

The FEA method could be build on the basics of force method in the building construction mechanics. That is the so called FEA method for stresses that is based on the principle of minimum for the full strain energy. The applications of this variant is more complex compared to the FEA method for the displacements.

The stated above variants of FEA methods are known as one-field ones, because they are based on the variation of only one field, either the field of displacements or the field of stresses. It should be noted that, there exist mixed and hybrid variants of the FEA method. These are the so called two-field methods. In the mixed variants, the variation is done with the 2 fields (of displacements and the stresses) in the investigated region. In the hybrid variants, one of the fields is in the region, and the other is in the boundary of the region.

All of the mentioned so far variants of FEA method are applicable only for problems where it is possible the defining a functional (f.e. strain energy). There are also variants of the FEA method free of that restriction. These are variants build on the bases of the methods of weighted remains, as for example the well known methods of Galjorkin, Bubnov-Galjorkin and Petrov-Galjorkin.

2.4.Principle of the virtual displacements

The term “system” in the mechanics is defined as the combination of physical structure and the applied loads on it (static boundary conditions). The configuration of a system is determined by the position of all parts in the structure. The fig.2.2 shows a system with exit configuration K_1 and deformed configuration K_2.

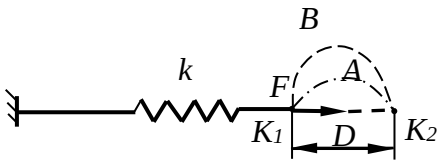
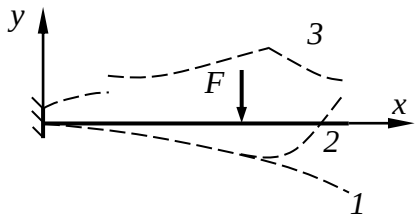


Fig.2.2

A system is called conservative if the work of the outer loads, so as the work of the inside forces do not depend on the path, traveled between K_1 and K_2. In an elastic type of a system the work of the external loads is equal to the strain energy, stored in the system.

A possible configuration after the deformation of the system is that, which satisfies the internal continuity and the kinematical boundary conditions. For example, on the fig.2.3 are shown the possible configurations 1 and 2, and the configuration 3 obviously do not satisfies the requirements for possible configuration.

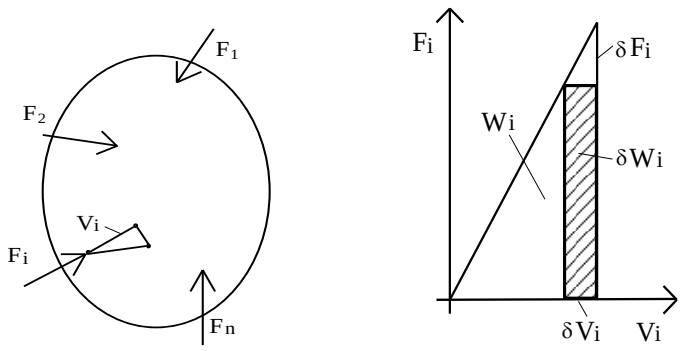


2.5. Deformational work of the external forces.

The deformational work done by the force F_i during the deforming an elastic body caused by the action of a system of self-balancing forces $F_I – F_n$ (fig.2.4. a) is

$$W_i = \frac{1}{2} F_i . v_i \qquad (2.1)$$

where v_i is the projected displacements of the point of application of the force from all acting loads on the body.



a) b)

fig. 2.4

the deformational work of all the forces is:

$$W = \frac{1}{2}\left(F_1 v_1 + F_2 v_2 + \ldots + F_i v_i + \ldots + F_n v_n\right) = \\ = \sum_1^n \frac{1}{2} F_i v_i = \frac{1}{2}\{v\}^T \{F\} \qquad (2.2)$$

where: $\{v\}^T = [v_1\ v_2\ \ldots v_n]$ is the transposed matrix of the displacements; $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ \ldots \\ F_n \end{Bmatrix}$ - column matrix of the forces

If there are volumetric forces acting on the body $\{R_v\} = \begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix}$ and distributed on surface *A* surface forces $\{p\} = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$, where $R_x, R_y, R_z, p_x, p_y, p_z$ are the components of the volumetric forces and the intensity of the surface distributed forces, for their deformational work could be written

$$W = \frac{1}{2}\left(\int_v \{u\}^T \{R_v\} dV + \int_A \{u\}^T \{p\} dA \right) \qquad (2.3)$$

In (2.3) $\{u\}^T = [u\ v\ w]$ is the transposed matrix of the displacements of an arbitrary point on the body.

If there is infinitely small increasing of the force $F_i - \delta F_i$ (фиг. 2.4, b) the increment of it's deformational work is:

$$\Delta W_i = F_i . \delta v_i + \frac{1}{2} \delta F_i . \delta v_i \qquad (2.4)$$

where δv_i is the increment of the projected displacement of the force from the change of the force.

The first additive is the main part of the increment and is called first variation or just variation of the deformational work.it is denoted with δW_i and

$$\delta W_i = F_i.\delta v_i. \tag{2.5}$$

the total increment for all the forces is:

$$\delta W = \{\delta v\}^T \{F\}. \tag{2.6}$$

If there are volumetric or surface forces we get:

$$\delta W = \int_v \{\delta u\}^T \{R_v\} dV + \int_A \{\delta u\}^T \{p\} dA. \tag{2.7}$$

In the upper relations the matrix $\{\delta u\}$ determines the actual displacements that correspond to the increase of the forces. If there are connections inside the body , the $\{\delta u\}$ are infinitely small displacements, which must satisfy as the continuity conditions inside the body, so as the conditions on the surface. These displacements are called virtual displacements.

2.6. Strain energy of the deformations

The specific strain energy stored in the body during it's deformation is

$$u_o = \frac{dU}{dV} = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + ... + \tau_{xy}\gamma_{xy} + ...\right) = \frac{1}{2}\{\varepsilon\}^T\{\sigma\}, \tag{2.8}$$

where: $\{\varepsilon\}^T = \lfloor \varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx} \rfloor$ is the transposed matrix of the deformations;

$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$ - column matrix of stresses.

According to (2.8) the strain energy in a given volume could be determined via

$$U = \int_V \frac{1}{2}\{\varepsilon\}^T\{\sigma\} dV. \tag{2.9}$$

If there is an infinitely small change of the stress $\delta\sigma_x$ (fig. 2.5) the deformation is changed too ε_x with $\delta\varepsilon_x$. The change in the specific strain energy is

$$\delta u_o(\sigma_x) = \sigma_x.\delta\varepsilon_x + \frac{1}{2}\delta\sigma_x.\delta\varepsilon_x. \tag{2.10}$$

where

$$\delta u_o(\sigma_x) = \sigma_x.\delta\varepsilon_x \tag{2.11}$$

is the variation of the specific strain energy.
For the full specific strain energy the variation is:

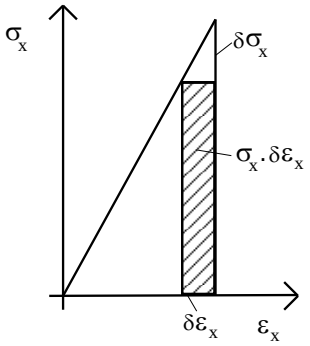
$$\delta u_o = \{\delta\varepsilon\}^T\{\sigma\}. \tag{2.12}$$

The variation of the strtain energy stored in a unity volume is:

$$dU = \delta u_o . dV . \tag{2.13}$$

the variation of the strain energy in the whole volume is:

$$\delta U = \int_v \{\delta\varepsilon\}^T \{\sigma\} dv . \tag{2.14}$$



Фиг. 2.5

If $\{\delta u\}$ are the virtual displacements, the $\{\delta\varepsilon\}$ will determine the possible deformations.

It is proved in the theory if elasticity that for a conservative system

$$\delta U = \delta W . \tag{2.15}$$

That is the definition of the principle of the virtual displacements. According to that principle, the work of the external forces for the virtual displacements is equal to the variation of the strain energy of the deformations.

2.7. Principle of stationarity of the full strain energy

One conservative system has strain energy that depends on the current configuration of the system , not on it's history or the path it has traveled to gain this configuration. The full strain energy is defined as

$$\Pi = U + \Omega , \tag{2.16}$$

where U is the strain energy of the deformations according to (2.9.), and Ω is the so called potential of the forces. The potential of the forces is the energy required ,roughly said, for putting back the forces it their start position before deforming of the system, in their points of application. So for the system of forces from fig.2.4. the potential is defined as:

$$\Omega = -\{v\}^T \{F\} . \tag{2.17}$$

The negative sign is there because the displacement is with opposite direction of the forces.For the variation of the potential of the forces we get:

$$\delta\Omega = -\{\delta v\}^T \{F\} . \tag{2.18}$$

The Principle of stationarity of the full strain energy is defined on the basis of the full strain energy. According to that principle, from all the possible configurations of one conservative system, the full strain energy is stationary with respect to small possible changes of the displacements, for that which meets the conditions for equilibrium,i.e.:

$$\delta\Pi = \delta U + \delta\Omega = 0 . \tag{2.19}$$

For a system with many degrees of freedom, as it is the discretised on small parts system in the FEA method, the full strain energy will be a function of the virtual displacements of the nodes $\Pi = \Pi(\Delta_1\ \Delta_2 \ldots \Delta_n)$. Then according to the Principle of stationarity of the full strain energy we could write:

$$d\Pi = \frac{\partial\Pi}{\partial\Delta_1} d\Delta_1 + \frac{\partial\Pi}{\partial\Delta_2} d\Delta_2 + \ldots + \frac{\partial\Pi}{\partial\Delta_n} d\Delta_n = 0 . \tag{2.20}$$

If we assume that only *dD1* is different from zero,it follows that $\frac{\partial\Pi}{\partial\Delta_1} = 0$. This way we get the system of equations: $\frac{\partial\Pi}{\partial\Delta_1} = 0$, $\frac{\partial\Pi}{\partial\Delta_2} = 0$, ..., $\frac{\partial\Pi}{\partial\Delta_i} = 0$, ... , *i=1, 2, ..., n* or

$$\left\{\frac{\partial\Pi}{\partial\Delta}\right\} = \{0\} . \tag{2.21}$$

The upper system of equations is solved with respect to the values of the displacements, that determine the static equilibrium of the system.

The full strain energy of an elastic body, loaded with conservative loads, acting on the volume V of the body and surface area A of it, is defined as:

$$\Pi = \int_V \left(\frac{1}{2} \{\varepsilon\}^T [E] \{\varepsilon\} - \{\varepsilon\}^T [E] \{\varepsilon_0\} + \{\varepsilon\}^T \{\sigma_0\} \right) dV - \\ - \int_V \{u\}^T \{R_V\} dV - \int_A \{u\}^T \{p\} dA - \{d\}^T \{P\} \quad . \qquad (2.22)$$

In (2.22) $\{\varepsilon_0\}$ and the $\{\sigma_0\}$ are the vectors of the starting deformations and stresses, as $\{\sigma_0\} = [E](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$, $\{R_V\}$ and $\{p\}$ are the vectors of the intensities of the volume and surface distributed forces and $\{P\}$ is a vector of nodal loads.

2.8. Displacements, deformations and stresses in an finite element.Functions of the shape.

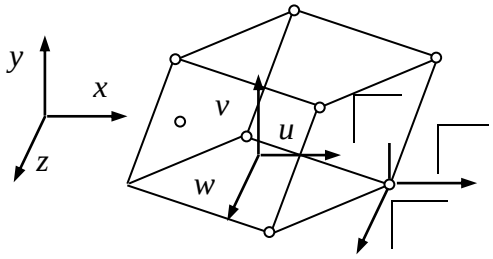


fig. 2.6

The nodal displacements in an finite element (fig.2.6) could be presented as a matrix in the form:

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ \dots \\ d_8 \end{Bmatrix}, \qquad (2.23)$$

As the elements $d_1, d_2, \dots, d_8$ are sub matrices, for example $\{d_1\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix}$, where u_1 is a nodal displacements of node 1 on the axis X from the chosen coordinate system, v_1 - the displacement on Y axis and so on.

An arbitrary point inside the element has displacements on the axis x,y and z that may be represented with a matrix as:

$$\{u\} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} . \qquad (2.24)$$

The displacements of that arbitrary point could be approximated functions of polynomials and then we can write:

$$\{u\} = [\Phi]\{a\}, \qquad (2.25)$$

where $[\Phi]$ is a square matrix, which elements are functions of the coordinates *x, y* and *z* of the point and their degrees, and $\{a\}$ is a column matrix of polynomial coefficients.
The nodal displacements could be determined on the basis of (2.25) using the nodal coordinates, namely:

$$\{d\} = [C]\{a\}, \tag{2.26}$$

where the matrix $[C]$ has elements defined by the nodal coordinates.
If in (2.26) we define

$$\{a\} = [C]^{-1}\{d\} \tag{2.27}$$

and substitute in (2.25) we get:

$$\{u\} = [\Phi][C]^{-1}\{d\} = [N]\{d\}. \tag{2.28}$$

The matrix $[N] = [\Phi][C]^{-1}$ consists of elements, functions of the coordinates of the point and the coordinates of the nodes. It is called matrix of the functions of the shape. In the common case $[N]$ is a square matrix as the number of the rows is equal to the components of the $\{u\}$, and the number of the columns is equal to the components of $\{d\}$ and it could be presented as:

$$[N] = [N_1\ N_2, ..., N_i, ...], \quad i = 1, 2, ..., n, \tag{2.29}$$

where *n* is the number of degrees of freedom in the nodes of the element.

The functions of the shape are called standard if they posses the following properties:

1. Every function N_i describes the change of the displacement in the region of the element when the displacement in node *i* is unity, and for the remaining nodes it is zero.
2. Every function N_i is a polynomial with a degree equal to the degree of the function, approximating the displacements of the element.

The deformations inside the element are determined by the displacements using the Cauchi's equatuions

$$\{\varepsilon\} = [D]\{u\}, \tag{2.30}$$

where $[D]$ is a differential matrix.
Taking into account (2.28) we can write:

$$\{\varepsilon\} = [D][N]\{d\} = [B]\{d\}, \tag{2.31}$$

where $[B] = [D][N]$ is a square matrix with number of the rows equal to the components of $\{\varepsilon\}$ and number of columns equal to the components of $\{d\}$. Taking into account (2.31) the matrix $[B]$ could be written as:

$$[B] = [B_1\ B_2\ ...B_i\ ...], \quad i = 1, 2, ..., n, \tag{2.32}$$

where

$$[B_i]=[D][N_i]=\begin{bmatrix} \partial N_i/\partial x & 0 & 0 \\ 0 & \partial N_i/\partial y & 0 \\ 0 & 0 & \partial N_i/\partial z \\ \partial N_i/\partial y & \partial N_i/\partial x & 0 \\ 0 & \partial N_i/\partial y & \partial N_i/\partial z \\ \partial N_i/\partial z & 0 & \partial N_i/\partial x \end{bmatrix}. \quad (2.33)$$

The stresses in the finite element are determined according to the Hook's Law

$$\{\sigma\}=[E]\{\varepsilon\}=[E][B]\{d\}. \quad (2.34)$$

2.9 Stiffness matrix of the finite element

2.9.1. Deriving the stiffness matrix on the basis of the principle of the virtual displacements.

If we investigate the equilibrium of a finite element, external forces for it are the forces from the stresses on the boundaries with the other finite elements (as a result from the interaction between them), volume forces and surface forces,if the finite element is on the surface of the body.

With arbitrary infinitely small increment of the nodal displacements $\{\delta d\}^T=\lfloor \delta d_1\, \delta d_2 \ldots \rfloor$, all the points from the element get an increment:

$$\{\delta u\}=[N]\{\delta d\}. \quad (2.25$$

The work of the external loads for the displacements $\{\delta u\}$ is δW_e , and the change of the potential energy is δU_e . According to the principle of the virtual displacements $\delta W_e=\delta U_e$.

All external loads (concentrated,surface,volumetric) are replaced by nodal forces

$\{r\}=\begin{Bmatrix} r_1 \\ r_2 \\ \ldots \end{Bmatrix}$, as r_1 is the nodal force in node *1* of the element r_2 - in node *2* and so on.

Because the external loads are replaced by nodal according to the equivalency of the deformational work, so :

$$\delta W_e=\{\delta d\}^T\{r\}. \quad (2.36)$$

The variation of the strain energy of the deformations is determined on the basis of the increments of the deformations themselves:

$$\{\delta\varepsilon\}=[D]\{\delta u\}=[D][N]\{\delta d\}, \quad (2.37)$$

or

$$\{\delta\varepsilon\}=[B]\{\delta d\}. \quad (2.38)$$

Then, taking into account (2.34) we can write:

$$\delta U_e=\int_V \{\delta\varepsilon\}^T\{\sigma\}dV=\int_V \{\delta d\}^T[B]^T[E][B]dV.\{d\}, \quad (2.39)$$

from where

$$\delta U_e = \{\delta d\}^T \left(\int_V [B]^T [E][B] dV \right) \{d\} = \{\delta d\}^T [k] \{d\}. \quad (2.40)$$

The equation

$$\int_V [B]^T [E][B] dV = [k] \quad (2.41)$$

is actually a square matrix with number of the rows and the columns equal to the number of the nodal parameters in the finite element.

From (2.36) and (2.40) we receive:

$$[k]\{d\} = \{r\}. \quad (2.42)$$

That is the relationship between the nodal forces and the nodal displacements.The matrix $[k]$ is called stiffness matrix of the finite element. Of the geometry of the finite element is known and the material properties are assigned, the matrix $[k]$ is completely defined with the choice of approximating functions for the displacements inside the element.

2.9.2. Deriving the stiffness matrix on the basis of the principle of the stationarity of the full strain energy

Taking into account that for an finite element the relations $\{u\} = [N]\{d\}$, $\{\varepsilon\} = [D]\{u\} = [D][N]\{d\} = [B]\{d\}$ are valid, for the full strain energy (2.22) we get:

$$\Pi = \frac{1}{2} \sum_1^n \{d\}_n^T [k]_n \{d\}_n - \sum_1^n \{d\}_n^T \{r\}_n - \{\Delta\}^T \{P\}. \quad (2.43)$$

where

$$[k] = \int_{V_e} [B]^T [E][B] dV \quad (2.44)$$

is the stiffness matrix of the finite element and the integrating is done on the volume of the FE, and

$$\{r\} = \int_{V_e} [B]^T [E]\{\varepsilon_0\} dV - \int_{V_e} [B]^T \{\sigma_0\} dV + \int_{V_e} [N]^T \{R_V\} dV + \int_{A_e} [N]^T \{p\} dA \quad (2.45)$$

is a vector of the nodal loads of the FE. A_e is the surface of the FE with the loads distributed on the surface, $\{P\}$ is a vector of the applied loads in the nodes and $\{\Delta\}$ is the vector of the nodal displacements for the whole system.

In (2.43) the addition is done for all the elements of the system and $\{d\}$ could be replaced by $\{\Delta\}$, if initially the matrices $[k]$ and $\{r\}$ are expanded to the dimension of the matrices for the whole system.After the addition we get:

$$\Pi = \frac{1}{2} \{\Delta\}^T [K] \{\Delta\} - \{\Delta\}^T \{R\}, \quad (2.46)$$

where

$$[K] = \sum_1^n [k]_n \text{ и } \{R\} = \{P\} + \sum_1^n \{r\}_n. \quad (2.47)$$

In (2.47) the addition means assembling of the elements of the matrices. The received in (2.46) full strain energy is a function of the nodal displacements $\{\Delta\}$.

According to the principle of the stationarity of the full strain energy

$$\left\{\frac{\partial \Pi}{\partial \Delta}\right\} = \{0\}, \tag{46}$$

that leads to

$$[K]\{\Delta\} = \{R\}. \tag{47}$$

2.9.3. Properties of the stiffness matrix

The stiffness matrices of the FEs $[k]$ and the global stiffness matrix for the system $[K]$ has the following important properties:

1. When we have linear-elastic behavior of the material,the matrices are symmetrical.
2. All the elements of the main diagonal of the matrices $[k]$ and $[K]$ are positive values. That is easily proved,if we assume that only the element $d_i \neq 0$, that means the related to that displacement load $r_i \neq 0$ therefore $k_{ii}d_i = r_i$. From the last equation we may see that if $k_{ii} < 0$,then r_I and d_I have different directions.

The global stiffness matrix is singularity matrix if the body (structure)is with wrong build connections or it is not supported at all.In that cases there is no solution of the system of equations. $[K]\{\Delta\} = \{R\}$. The connections must be available in order to restrict all possible displacements of the body as rigid non deformable one or from the displacements caused by the deformations of the structure (body).

2.10. Deriving the vector of the loads

If there are, volume distributed over the surface, forces applied to a FE or non-nodal concentrated forces, the change of the deformational work for virtual displacements $\{\delta u\}$ of the points of the body will be:

$$\delta W = \int_V \{\delta u\}^T \{R_V\} dV + \int_A \{\delta u\}^T \{p\} dA + \{\delta u\}^T \{P\}. \tag{2.48}$$

Considering that (2.28) and $\{\delta u\}^T = \{\delta d\}^T [N]^T$, (2.48) could be written in the following manner:

$$\delta W = \{d\}^T \left(\int_V [N]^T \{R_V\} dV + \int_A [N]^T \{p\} dA + [N]^T \{P\} \right). \tag{2.49}$$

When transforming non=nodal loads into nodal ones the deformational work could be defined by the relations"

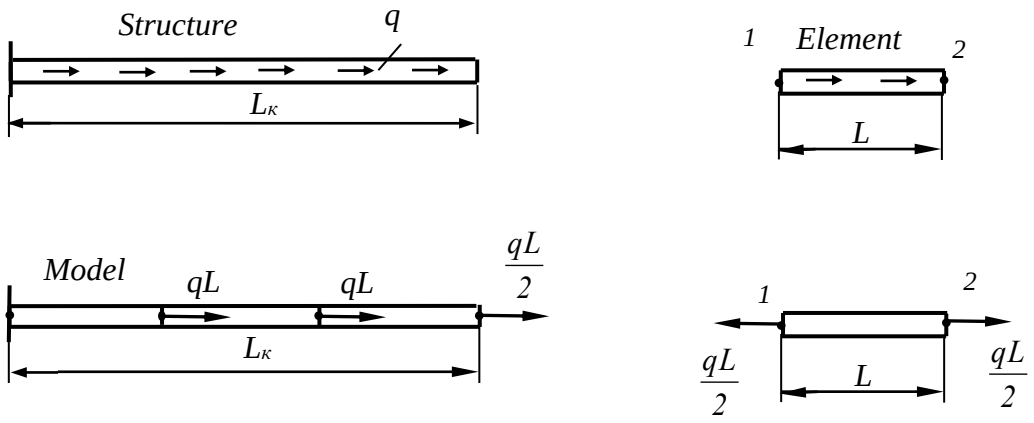
$$\delta W = \{\delta d\}^T \{r\}, \tag{2.50}$$

where $\{r\}$ is a vector of the nodal loads. If for the transforming we accept deformational equivalency, then after the equalizing of (2.49) and (2.50) we get:

$$\{r\} = \int_V [N]^T \{R_V\} dV + \int_A [N]^T \{p\} dA + [N]^T [P]. \tag{2.51}$$

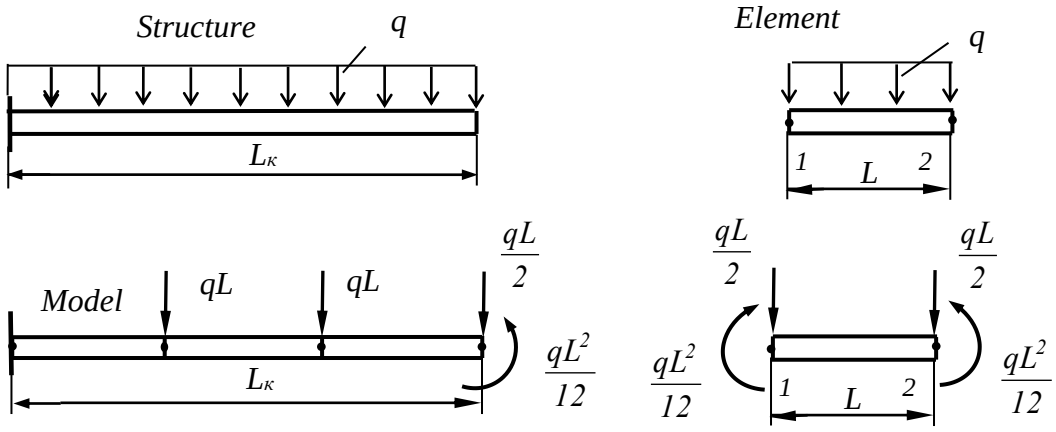
In order not to transform the concentrated loads, in creating the mesh of finite elements, the points of application of there loads are considered to be nodes.

Examples of transforming loads in truss and beam structures are shown respectively on figures 2.7 and 2.8.



Фиг. 2.7

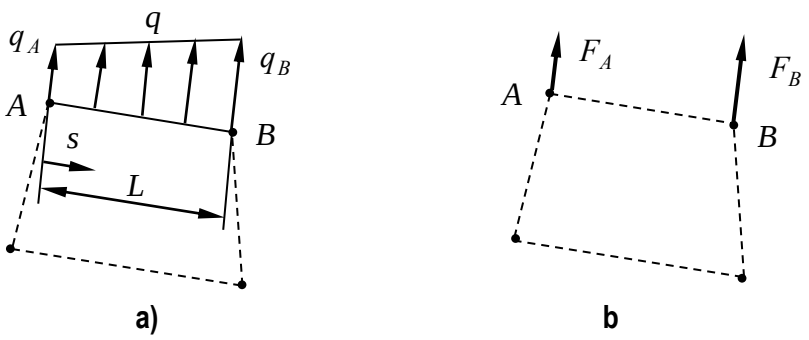
On fig. 2.7 the model consists of 3 elements, in the nodes of which, the forces are derived after transforming the distributed load.



Фиг. 2.8

In transforming the distributed load in the beam model from fig. 2.8 a moment arises in the nodes.

In the plane type of problems, if the direction of the intensity with respect to the boundary of the element , it can be dissolved in the perpendicular and the tangential direction of the boundary. In transforming the loads to the nodes, the following pictures are obtained below in figures 2.9 and10.



a) **b**

fig

According to (2.51), in transforming of the linearly distributed load from fig. 2.9, for the nodal forces we get: $\begin{Bmatrix} F_A \\ F_B \end{Bmatrix} = \frac{L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} q_A \\ q_B \end{Bmatrix}$.

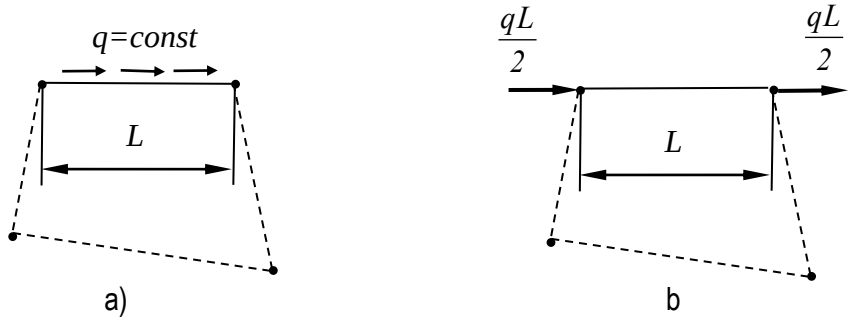


fig. 2.10

Fig. 2.11 shows the transformation of square law distributed load fir. 2.11, a). According to (2.51) for the nodal forces (fig. 2.11, b) we get: $\begin{Bmatrix} F_A \\ F_B \\ F_C \end{Bmatrix} = \frac{L}{30}\begin{bmatrix} 3 & 3 & -1 \\ 2 & 15 & 2 \\ -1 & 2 & 4 \end{bmatrix}\begin{Bmatrix} q_A \\ q_B \\ q_C \end{Bmatrix}$.

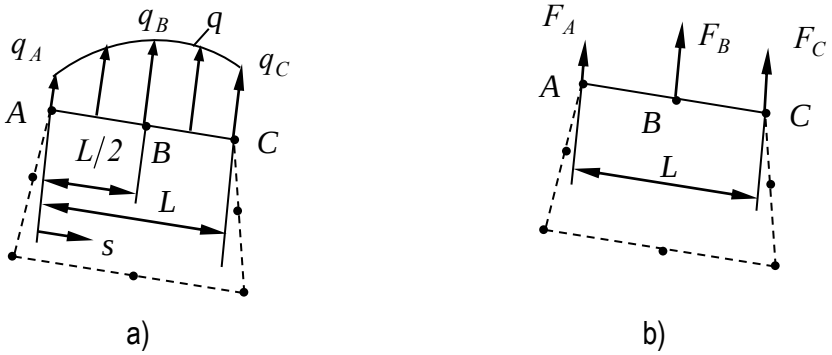


fig. 2.11

Fig.2.12 shows the transformation of volume distributed forces from the own weight of the body. While in the case of 3-angle 3-node(fig. 2.12, a) and the 4-angle 4-node (2.12, c) elements the obtained configuration is expected, in the case of 6-node 3-angle (2. 12, b) and the 8-node 4-angle (2. 12, d) elements the configuration is unexpected.

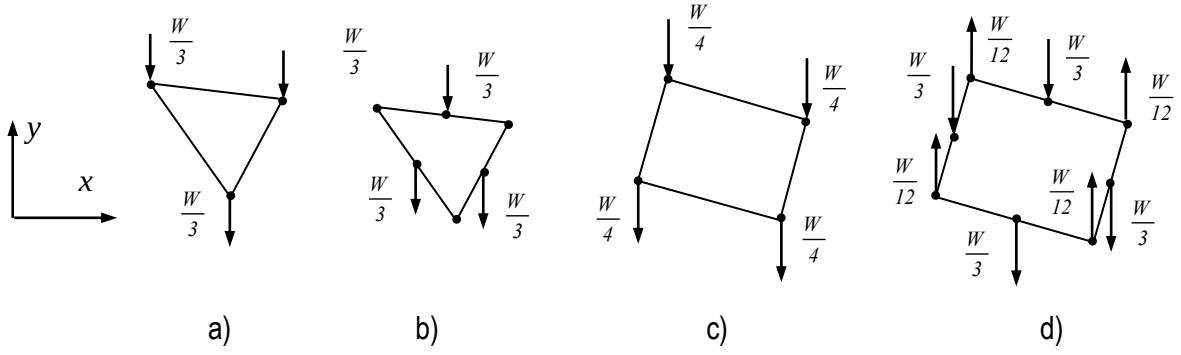
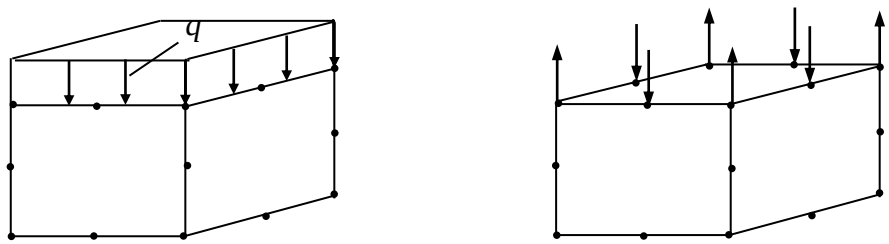


fig. 2.12

The orientation of the coordinate system in of no interest, but in order to receive the deformational equivalency of the loads in the 4-angle element, it is necessary the element to be rectangle.

The transforming of distributed pressure on a hexahedron 20 node element is shown on fig. 2.13.



Фиг. 2.13

The forces in the angle nodes are $\frac{pA}{12}$, and in the intermediate nodes $\frac{pA}{3}$, where *A* is the area of the wall under the pressure. The wall under pressure needs to be a rectangle, so that we will have deformational equivalency of the loads.

2.11. Assembling the global stiffness matrix of the system

Nevertheless, the nodal parameters in (2.42) are the only unknown quantities, as we already mentioned, its defining could not be done out of these equations, because the conditions for the equilibrium of the nodes are not taken into account. The static equilibrium conditions for the nodes could be written after the relations (2.42) are transformed from the local coordinate system (in which its deriving is easier) in the global one. After the application of the static equilibrium conditions, we get an global system of equations, that can be written in matrix form as follows:

$$[K]\{\Delta\} = \{R\}\ , \tag{2.52}$$

where $\{R\}$ is a vector of the nodal loads, containing all the nodes of the system, $\{\Delta\}$ is a vector of the nodal parameters of the whole system and $[K]$ is the so called global stiffness matrix of the system.

The global stiffness matrix of the system is a square symmetric matrix with number of rows and columns equal to the number of the nodes:

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ . & . & . & \\ . & . & . & \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix}. \tag{2.53}$$

In the general case the elements of that matrix could be sub matrices with number of the elements, equal to the number of the nodal degrees of freedom. For a model sub matrix we have:

$$[K_{rs}] = \sum_e [k_{rs}], \tag{2.54}$$

as $\sum_e [k_{rs}]$ means summing on all elements containing nodes r and s at one and the same time.

Fig. 2.14 shows an example for assembling of 2D region.The elements are 3-angle,3-node with 2 degrees of freedom per node.

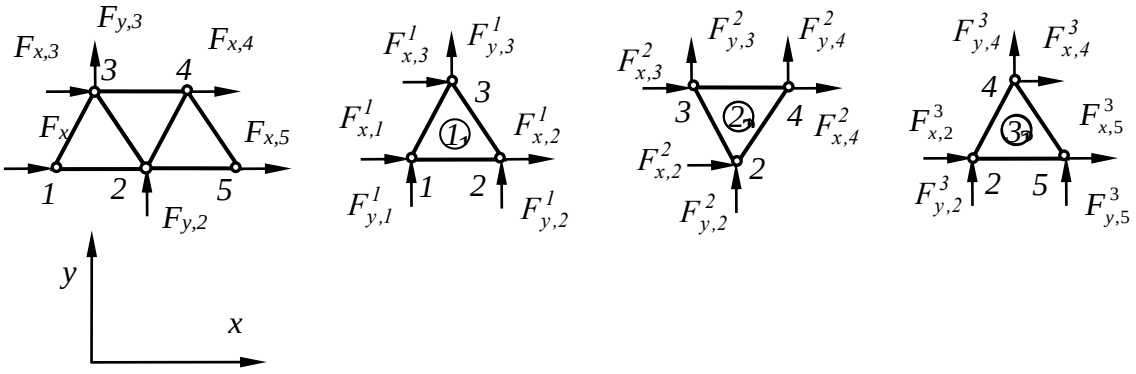


fig.14

From the equilibrium conditions for the nodes we can write:

For node 1 $F^1_{x,1} = F_{x,1}$, $F^1_{y,1} = 0$. For node 2 $\begin{matrix} F^1_{x,2} + F^2_{x,2} + F^3_{x,2} = 0 \\ F^1_{y,2} + F^2_{y,2} + F^3_{y,2} = F_{y,2} \end{matrix}$.

For node 3 $\begin{matrix} F^1_{x,3} + F^2_{x,3} = F_{x,3} \\ F^1_{y,3} + F^2_{y,3} = F_{y,3} \end{matrix}$. For node 4 $\begin{matrix} F^2_{x,4} + F^3_{x,4} = F_{x,4} \\ F^2_{y,4} + F^3_{y,4} = 0 \end{matrix}$.

For node 5 $\begin{matrix} F^3_{x,5} = F_{x,5} \\ F^3_{y,5} = 0 \end{matrix}$.

For the 3 FEs we can write:

element 1 $\begin{bmatrix} k^1_{ii} & k^1_{ij} & k^1_{im} \\ k^1_{ji} & k^1_{jj} & k^1_{jm} \\ k^1_{mi} & k^1_{mj} & k^1_{mm} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F^1_1 \\ F^1_2 \\ F^1_3 \end{Bmatrix}$, element 2 $\begin{bmatrix} k^2_{ii} & k^2_{ij} & k^2_{im} \\ k^2_{ji} & k^2_{jj} & k^2_{jm} \\ k^2_{mi} & k^2_{mj} & k^2_{mm} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_4 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F^2_2 \\ F^2_4 \\ F^2_3 \end{Bmatrix}$

and element 3 $\begin{bmatrix} k^3_{ii} & k^3_{ij} & k^3_{im} \\ k^3_{ji} & k^3_{jj} & k^3_{jm} \\ k^3_{mi} & k^3_{mj} & k^3_{mm} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_5 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F^3_2 \\ F^3_5 \\ F^3_4 \end{Bmatrix}$.

The index in the upper part is for the number of the element. The elements of all matrices are sub matrices with 2 elements for the components on x and y axis.

After we solve the matrices and use the static equilibrium conditions we get:

$$k^1_{ii} d_1 + k^1_{ij} d_2 + k^1_{im} d_3 = F^1_1 = \begin{Bmatrix} F_{x,1} \\ 0 \end{Bmatrix} = r_1$$

$$k^1_{ji} d_1 + \left(k^1_{jj} + k^2_{ii} + k^3_{ii}\right) d_2 + \left(k^1_{jm} + k^2_{im}\right) d_3 + \left(k^2_{ij} + k^3_{im}\right) d_4 + k^3_{ij} d_5 = F^1_2 + F^2_2 + F^3_2 = \begin{Bmatrix} 0 \\ F_{y,2} \end{Bmatrix} = r_2$$

$$k^1_{mi} d_1 + \left(k^1_{mj} + k^2_{mi}\right) d_2 + \left(k^1_{mm} + k^2_{mm}\right) d_3 + k^2_{mj} d_4 = F^1_3 + F^2_3 = \begin{Bmatrix} F_{x,3} \\ F_{y,3} \end{Bmatrix} = r_3$$

$$\left(k_{ji}^2+k_{mi}^3\right)d_2+k_{jm}^2d_3+\left(k_{jj}^2+k_{mm}^3\right)d_4+k_{mj}^3d_5=F_4^2+F_4^3=\begin{Bmatrix}F_{x,4}\\0\end{Bmatrix}=r_4$$

$$k_{ji}^3d_2+k_{jm}^3d_4+k_{jj}^3d_5=F_5^4=\begin{Bmatrix}F_{x,5}\\0\end{Bmatrix}=r_5.$$

The equations written above could be written in matrix form as follows:

$$\begin{bmatrix}k_{ii}^1 & k_{ij}^1 & k_{im}^1 & 0 & 0\\ k_{ji}^1 & \left(k_{jj}^1+k_{ii}^2+k_{ii}^3\right) & \left(k_{jm}^1+k_{im}^2\right) & \left(k_{ij}^2+k_{im}^3\right) & k_{ij}^3\\ k_{mi}^1 & \left(k_{mj}^1+k_{mi}^2\right) & \left(k_{mm}^1+k_{mm}^2\right) & k_{mj}^2 & 0\\ 0 & \left(k_{ji}^2+k_{mi}^3\right) & k_{jm}^2 & \left(k_{jj}^2+k_{mm}^3\right) & k_{mj}^3\\ 0 & k_{ji}^3 & 0 & k_{jm}^3 & k_{jj}^3\end{bmatrix}\begin{Bmatrix}d_1\\d_2\\d_3\\d_4\\d_5\end{Bmatrix}=\begin{Bmatrix}r_1\\r_2\\r_3\\r_4\\r_5\end{Bmatrix}.$$

The result could be easily found as the individual matrices of the elements are expanded from 3x3 and 5x5 and after that we add them.

2.12. Other approaches in FEA method.

As already have been noted, there are variants of the FEA method built on the method of the weighted residiums. The advantage here is that the basic equations in the FESA could be obtained directly from the differential equations of the problem. That means, it is not necessary to look for the respective functional ,equivalent to known differential equations. On the other hand, these variants can be applied for problems that do not have a functional at all, too.

We solve the problem for approximate solution of the differential equations

$$Du-f=0 \text{ in the region } V\, Bu-g=0 \text{ on the boundary } S \text{ from } V, \qquad (2.55)$$

where *u(x)* is unknown function, satisfying the equations (2.55), *f(x)* and *g(x)* are known functions, given in the region *V* and the boundary *S. D* and *B* are differential operators.

In the general case the exact solution $u=u(x)$ is unknown and we are looking for approximated one $\bar{u}=\bar{u}(x)$. Usually $\bar{u}$ is a polynomial with unknown coefficients $a_1, a_2, ..., a_n$. The coefficients are defined so, that the function $\bar{u}=\bar{u}(a,x)$ is close to the *u(x)*.

If $\bar{u}(x)$ is approximated solution of the equations (2.55), it could be written:

$$D\bar{u}-f=R_1(a,x),\quad B\bar{u}-g=R_2(a,x), \qquad (2.56)$$

where $R_1(x)$ and $R_2(x)$ are the so called residium functions.

The best approximated solution could be received from the condition:

$$\int_V R_1dV+\int_S R_2dS=min\ . \qquad (2.57)$$

The residium functions could be set to zero for some values of x , but they are not zeros for all values of x, because it could be the case only in the exact solution, i.e. when $u\equiv\bar{u}$. Therefore the smaller the residiums, the better the approximation is. Small residuims could be found in different ways, each of which leads the problem to algebraic equations, according to which one could define n number if coefficients.

The basic relation,that leads to the methods of the weighted residiums has the look

The basic relation,that leads to the methods of the weighted residiums has the look

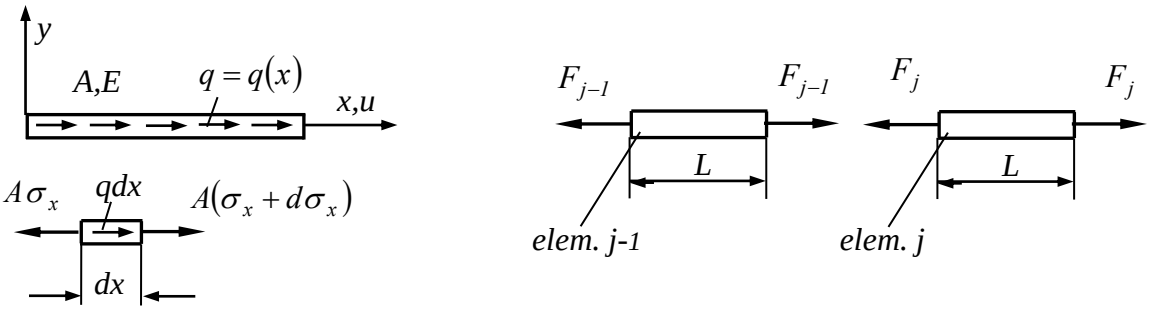
$$\int_V R_1W_1(x)dV+\int_S R_2W_2(x)dS=0, \qquad (2.58)$$

where $W_1(x)$ and $W_2(x)$ are suitably chosen functions, called weight functions. The difference in the methods is in the choice of the weight functions. The point of (2.58) is that, for the region under investigation, the weighted residuims have approximated value zero or the multiplications $W_i(x)R_i$, $i = 1, 2, 3$ have zero approximated value.

The most popular of the weighted residuims methods is the method of Galjorkin. The essential part of that method is the choice of weigh functions $W_i = W_i(x)$ and then setting the approximated weighted residiums to zero, with respect to the region under investigation.

$$R_i = \int_V W_i(x) R_1(a,x) dV = 0, \quad i = 1, 2, ..., n. \tag{2.59}$$

In that method the weight functions W_i are defined as derivatives of the approximating function with respect to the coefficients a_i, i.e. $W_i = \frac{\partial \bar{u}}{\partial a_i}$. One presentation of FESA with that method is done with the shown on the fig. 2.15, a truss. The truss is separated on 2 elements (fig. 2.15, b).



a) b)

fig 2.15

the differential equation for the equilibrium of the element from fig. 2.15, a is

$$A\frac{\partial \sigma_x}{\partial x} + q = 0. \tag{2.60}$$

because $\sigma_x = E\varepsilon_x = E\frac{\partial u}{\partial x}$, then the equation (2.60) could be written in the form:

$$AE\frac{\partial^2 u}{\partial x^2} + q = 0. \tag{2.61}$$

For the end of the element from fig. 2.15, b it can be written the static boundary condition

$$AE\frac{\partial^2 u}{\partial x^2} - F = 0, \tag{2.62}$$

as the right hand side of the truss *F=0.* The geometric boundary conditions are defined from the reccomended values of the displacements.

For the 2 finite elements(fig. 2.15, b) the field of displacements is approximated by the already know way

$$\bar{u} = [N]\{d\}, \text{ where } [N] = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right], \quad \{d\}^T = [u_1 \quad u_2]. \tag{2.63}$$

The weight functions according to the method of Galjorkin ar edefined by the equations:

$$W_i = \frac{\partial \bar{u}}{\partial d_i} = N_i, \text{ where } N_1 = (L - x)/L, \quad N_2 = x/L. \tag{2.64}$$

According to (2.59) now we can write:

$$\sum_{j=1}^{n_{en}} \int_0^L N_i \left(AE \frac{\partial^2 \bar{u}}{\partial x^2} + q \right) dx = 0 . \qquad (2.65)$$

If *E*=*const* for one element (2.65) can be integrated by parts, thus getting the result:

$$\int_0^L N_i AE \frac{\partial^2 \bar{u}}{\partial x^2} dx = \left| N_i AE \frac{\partial \bar{u}}{\partial x} \right|_0^L - \int_0^L \frac{\partial N_i}{\partial x} AE \frac{\partial \bar{u}}{\partial x} dx . \qquad (2.66)$$

Considering the static boundary condition (2.62) and (2.66) from (2.65) we get

$$\sum_{j=1}^{n_{en}} \int_0^L \left(-\frac{\partial N_i}{\partial x} AE \frac{\partial \bar{u}}{\partial x} + N_i q \right) dx + \sum_{j=1}^{n_{en}} \left| N_i F \right|_0^L = 0 \qquad (2.67)$$

If we note $[B] = \left[\frac{\partial N}{\partial x} \right]$ and considering (2.63), (2.67) could be written as:

$$\sum_{j=1}^{n_{en}} [B]^T AE[B] dx \{d\} = \sum_{j=1}^{n_{en}} \int_0^L [N]^T q dx + \sum_{j=1}^{n_{en}} \left| [N]^T F \right|_0^L . \qquad (2.68)$$

In (2.68) $\sum_{j=1}^{n_{en}} [B]^T AE[B] dx \{d\}$ obviously leads to $[K]\{D\}$, and $\sum_{j=1}^{n_{en}} \int_0^L [N]^T q dx$ - the elements of the matrix of the nodal loads from the derived non-nodal loads $\{R\}$. In assembling the system, the equation $\sum_{j=1}^{n_{en}} \left| [N]^T F \right|_0^L$ in (2.68) gives the matrix $\{P\}$ of the concentrated nodal loads.