Chapter 1

BASIC EQUATIONS IN THE THEORY OF ELASTICITY

1.1. Introduction. Basic principles in the Theory of elasticity.

Despite, theory of elasticity has been developed for more than 50 years the number of analytically solved problems is fairly low. Well developed methods used in "Strength of Materials" and Mechanics, can be applied only for simple constructions and in fact cannot be used for determination of stresses and deformation in complex real constructions. Unlike "Strength of Materials" and "Mechanics", elasticity theory allows obtaining more accurate results and more over gives solutions of numerous problems, which can be solved only by methods of this science. The equations of elasticity theory are the bases of every method for calculation of strength, including the method of Finite Elements.

Elasticity Theory is based on theoretical model of rigid body having the ability to be deformed. Using the so called working hypotheses, the model retains only the basic properties of a real body.

The main premises used for developing theoretical model of rigid deformable body are as follows:

1. **Perfect Elasticity.** It is presumed, that the arising deformations in the body after loading are entirely elastic, i.e. they disappear after load relief. This means when the body is loaded, the whole work done by the external loads is transformed into potential energy of deformations. After unloading, whole energy is retrieved without any energy losses in accordance to the first law of thermodynamics. In this way, property "perfect elasticity" can be characterized by consistent relationship between stresses and deformations in a given temperature. It should be noted, that most of the construction materials posses properties close to perfect elasticity and can be considered as ones.
2. **Linear relationship between stresses and deformations.**
3. **Homogeneity and Isotropy.** The premise isotropy means that for a small vicinity of a given particle of a body, the deformable body's properties are one and the same in all directions. If this is true for all particles of a body, it is homogenous. There are bodies, which are isotropic relative to a certain properties and anisotropic relative to other properties. A great interest related to the homogeneity and the isotropy are the dependencies between the stresses and the strain in the body. Construction materials which are cooled under normal conditions can be considered as homogenous and isotropic with negligible errors, despite singularity observed on a micro level.
4. **Initial Dimensions Principle.** It is assumed that an elastic body undergoes small deformations relative to its dimensions. This premise makes elasticity theory's equations much simpler because it is possible to neglect small quantities of higher order.
5. **Continuity of the Structure.** This premise allows considering the quantities characterizing strength and strain states in vicinity of a given particle as continuous and arbitrarily differentiable functions of coordinates. This premise was introduced by Cauchy and the continuity has not absolute but relative meaning because of the introduced by him average values of stresses and strains.
6. **Natural Strength State Principle.** According this premise, the stresses in a rigid deformable body are equal to zero, when it is not subjected to external forces or temperature changes. Elasticity Theory uses only stresses due to deformation of a body.
7. **Sen-Venan Principle.** It states that if a small surface area of a rigid body is subjected to a self-balancing system of forces, there arises, stresses which decrease rapidly as the distance increase. According the same principle, strength and strain states of particles separated on a bigger than the biggest dimension of the loaded section distance, does not depend on the way the forces are applied.
8. **The Laws of Static and Dynamic are Applicable for Every Elementary Volume of the Body.** Listed premises give the opportunity to solve a wide range of problems encountered in the practice. Accuracy and validity in most cases are confirmed by experiments. Results mismatching with the experiment data show that there is something, in all these premises, to be specified more precisely.

1.2. Basic notations and concepts.

The coordinate system of a body under investigation is *x, y, z*. With R_x, R_y and R_z are noted the components of the volume forces relative to unit volume, and with p_x, p_y and p_z - the components of the intensity of surface distributed forces. The matrix form of notation is as follows

$$\{R_V\}^T = [R_x \quad R_y \quad R_z] \text{ and } \{p\}^T = [p_x \quad p_y \quad p_z]. \tag{1.1}$$

For the detached body's elementary volume in vicinity of particle (fig. 1.1), the stress tensor can be written in matrix form as follows.

$$\{\sigma\}^T = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}]. \tag{1.2}$$

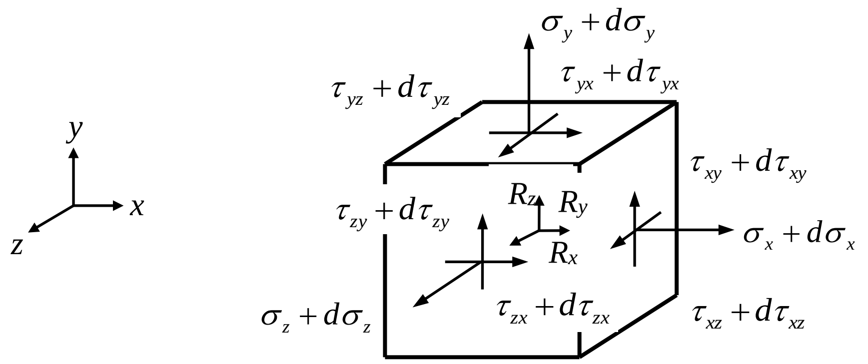

Fig. 1.1

First index of the tangential stress indicates the normal, and second – the direction. For the increase of stresses, it can be written

$$d\sigma_x(x,y,z) = \frac{\partial \sigma_x}{\partial x} dx, \ldots, d\tau_{xy}(x,y,z) = \frac{\partial \tau_{xy}}{\partial x} dx, \ldots \ . \tag{1.3}$$

The strain tensor can be written in matrix form as follows

$$\{\varepsilon\}^T = [\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{zx}]. \tag{1.4}$$

The displacements, in the vicinity of the particle, in *x, y* and *z* direction can be written in matrix form as follows

$$\{u\}^T = [u(x,y,z) \quad v(x,y,z) \quad w(x,y,z)]. \tag{1.5}$$

1.3. Basic Equations in the Theory of Elasticity

1.3.1. Differential Equations of Equilibrium

These equations are derived based on the static equilibrium conditions of the elementary parallelepiped shown in Fig. 1.1. If ρ is the mass density, taking into account the body forces R_x, R_y and R_z, it can be written

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + R_x = \rho \frac{\partial^2 u}{\partial t^2} \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + R_y = \rho \frac{\partial^2 v}{\partial t^2} \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + R_z = \rho \frac{\partial^2 w}{\partial t^2} \ .
\end{aligned} \tag{1.6}$$

The equations (1.6) are known as differential equations of equilibrium. From the static equilibrium conditions in the vicinity of a point (the element shown on Fig. 1.2) we obtain so called conditions of equilibrium at the surface.

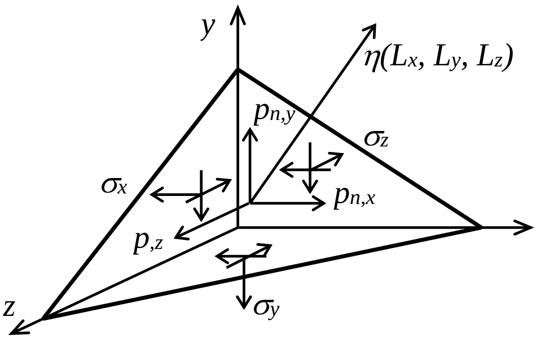

Fig. 1.2

If the direction cosine vector of the normal to the plane is denoted by $\{L\}^T = \left[L_x \ L_y \ L_z\right]$, from the equilibrium static conditions can be obtained

$$
\begin{aligned}
p_{n,x} &= \sigma_x L_x + \tau_{yx} L_y + \tau_{zx} L_z \\
p_{n,y} &= \tau_{xy} L_x + \sigma_y L_y + \tau_{zy} L_z \\
p_{n,z} &= \tau_{xz} L_x + \tau_{yz} L_y + \sigma_z L_z \ .
\end{aligned}
\tag{1.7}
$$

Equations (1.7) give the relationship between the intensity of the forces distributed on the surface and the stresses near the surface. These equations can be written in matrix form as follows

$$\{p\} = [T_\sigma]\{L\}, \tag{1.8}$$

$\left[T_\sigma\right]$ is the stress tensor.

1.3.2. Strain-displacement Relations

These equations give relation between the displacement components and the strain components. In case of 2D strain state (fig. 1.3)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\,, \tag{1.9}$$

and for the relative linear deformation ε_x can be written

$$\varepsilon_x = \frac{A' B' - AB}{AB}\,, \tag{1.10}$$

consequently

$$A' B' = (1 + \varepsilon_x)AB = (1 + \varepsilon_x)dx\,. \tag{1.11}$$

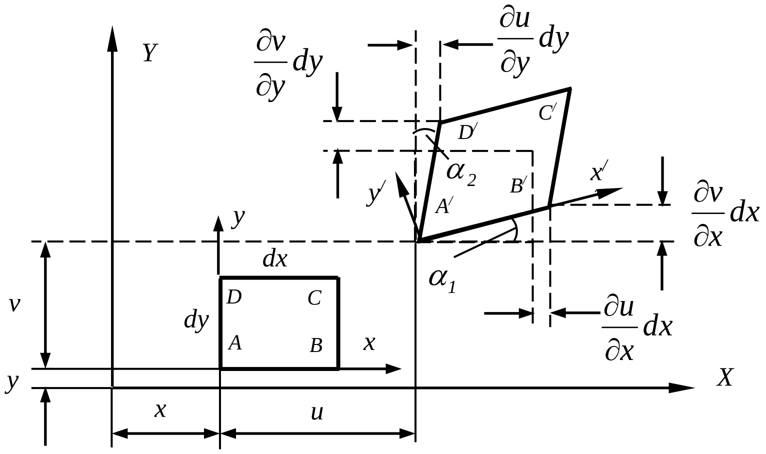

Fig. 1.3

According fig. 1.3

$$\left(A'B'\right)^2 = \left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2 . \tag{1.12}$$

After substituting of (1.11) in (1.12) is obtained

$$\left(1+\varepsilon_x\right)^2\left(dx\right)^2 = \left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2 , \tag{1.13}$$

consequently it is obtained

$$\varepsilon_x^2 + 2\varepsilon_x = 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 . \tag{1.14}$$

In case of small deformations and displacements $\varepsilon^2{}_x$, the squares of the derivatives with respect to *u* and *v* can be neglected, so the relationship between them for 3D case are determined by relations known as Cauchy's equations.

$$\begin{gathered}\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} .\end{gathered} \tag{1.15}$$

In case of small deformations and large displacements, ε_x^2 can be neglected and (1.14) it transforms to

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]. \tag{1.16}$$

In case of 3D problem with small deformations and large displacements, the relations between deformations and displacements are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right], \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial y} \, . \quad (1.17)$$

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right], \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}$$

In (1.17) the deformations are defined in the original coordinate system and are known as Green-LaGrange's equations.

Equations (1.15) can be written in matrix form as follows

$$\{\varepsilon\} = [D]\{u\}, \text{ where } [D] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad (1.18)$$

is so called differential matrix

If the displacements *u, v* and *w* are eliminated from equation (1.18) it is obtained the Sen-Venan's equations, known as compatibility (continuity) equations.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}; \quad \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right) = 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}, \quad \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} \quad (1.19)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial x \partial z}, \quad \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right) = 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} \, .$$

1.3.3. Physical Relationships

The following are equations, relating stresses and deformations, known as Hook's law. For isotropic body these relations are as follows

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right], \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_z + \sigma_x\right)\right], \quad \gamma_{yz} = \frac{1}{G}\tau_{yz} \quad (1.20)$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right], \quad \gamma_{zx} = \frac{1}{G}\tau_{zx} \, ,$$

where E is Young's modulus, G – sliding modulus and μ - Poisson's coefficient , which are related by $E = 2(1+\mu)G$. Equations (1.20) can be written in matrix form as follows

$$\{\varepsilon\} = [H]\{\sigma\}, \text{ where } [H] = \frac{1}{E}\begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix}. \qquad (1.21)$$

Equations (1.21) can be solved towards stresses and Hook's law to be expressed in Lampe's form

$$\begin{aligned} &\sigma_x = 2G\varepsilon_x + \lambda\theta, \quad \sigma_y = 2G\varepsilon_y + \lambda\theta, \quad \sigma_z = 2G\varepsilon_z + \lambda\theta \\ &\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx} \end{aligned}, \qquad (1.22)$$

where $\lambda = \frac{2\mu G}{1-2\mu} = \frac{\mu E}{(1+\mu)(1-2\mu)}$ is so called Lampe's parameter. The above equations can be written in matrix form as follows

$$\{\sigma\} = [E]\{\varepsilon\}, \qquad (1.23)$$

where

$$[E] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}. \qquad (1.24)$$

Matrix $[E]$ is the inverse of matrix $[H]$.

The equations (1.6), (1.15), (1.20) or (1.22) form a system of 15 equations, which contains 6 components of the stresses, 6 components of the deformations (strains) and 3 components of the displacements.

The solution of that problem can be obtained either for the displacements or for the stresses. By solving it towards displacements, the system of equations is reduced to three equations towards the functions $u(x,y,z), v(x,y,z)$ and $w(z,y,z)$, which are assumed to be the main unknown functions. To do this, stresses in the differential equations (1.6) for equilibrium are expressed according Hook's law (1.22) by means of the deformations. The equations obtained are substituted in (1.15). So, we get the Lampe's equations.

$$\begin{aligned} &G\nabla^2 u + (G+\lambda)\frac{\partial\theta}{\partial x} + R_x = 0 \\ &G\nabla^2 v + (G+\lambda)\frac{\partial\theta}{\partial y} + R_y = 0 \\ &G\nabla^2 w + (G+\lambda)\frac{\partial\theta}{\partial z} + R_z = 0, \end{aligned} \qquad (1.25)$$

where $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ и $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The boundary conditions should be included into these equations. When the body's surface displacement is given, the boundary conditions (called geometric or kinematical) should be satisfied by the functions $u(x,y,z), v(x,y,z)$ and $w(z,y,z)$. In case of given static boundary conditions (these are applied surface loads) it is necessary to satisfy the equations (1.7).

Solution towards stresses means, to assume the stress functions to be the main unknown functions. For that purpose, the equations (1.6) are used first, combined with the compatibility strain equations (1.19), in which the deformations should be expressed by the Hook's law for the stresses. The obtained equations are known as Beltrami-Michel's equations.

1.4. Two Dimensional Problems in Elasticity Theory

In practice, there is a wide range of problems in which the deformations and stresses depend only on two variables (for example *x* and *y*). These problems are also known as 2D or plane problems. Unlike 3D problems, in 2D problems the use of mathematics is much simpler. Although, in the reality such problems rarely could be observed, in many practical cases, the error coming out from the solution of a 3D problem considered as a 2D case is relatively small, so the calculations and the results can be accepted as valid. In elasticity theory are known two types of plane problems, 2D stress state and 2D strain state.

1.4.1. Two Dimensional Stress State

Typical 2D problem is shown on fig. 1.4. A disk loaded parallel to the middle plane, and the load is distributed uniformly along z axis.

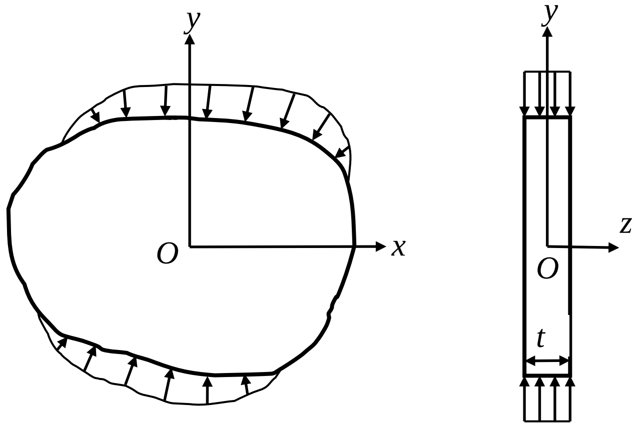

Fig. 1.4

If the disk thickness is much smaller than the other dimensions of the disk, it can be assumed, in case of loading as depicted on the figure above that stresses σ_z, τ_{zx} and τ_{zy} are equal zero along the periphery of the disk. Moreover, in case of the chosen above coordinate system O*xy* the points lying on periphery of the middle plane do not move in *z* direction. We can also assume that stresses different from zero σ_x, σ_y and τ_{xy} do not depend on *z*, because of the very small disk thickness. The described stress state so far, called generalized plane stress state, at first glance cannot be achieved in practice, but as mentioned before, the resultant error is small, and the mathematical model is significantly simplified.

Taking into account the previous consideration we can write

$$\begin{aligned} &\sigma_x = f_1(x,y), \quad \sigma_y = f_2(x,y), \quad \tau_{xy} = f_3(x,y) \\ &\sigma_z = 0, \quad \tau_{zx} = 0, \quad \tau_{zy} = o \end{aligned} \qquad (1.26)$$

These stresses should satisfy the equilibrium differential equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + R_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + R_y = 0 \ , \tag{1.27}$$

if the inertia forces are negligible. The surface equations are

$$p_{n,x} = \sigma_x L_x + \tau_{yx} L_y$$
$$p_{n,y} = \tau_{xy} L_x + \sigma_y L_y \ . \tag{1.28}$$

Cauchy's equations acquire the following form

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{1.29}$$

The above equations can be written in matrix form as in (1.18), but the differential matrix is

$$[D] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \ . \tag{1.30}$$

So the Hook's law is

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_z\right), \quad \gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\mu)}{E}\tau_{xy}$$
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right), \quad \gamma_{yz} = 0 \tag{1.31}$$
$$\varepsilon_z = \frac{-\mu}{E}\left(\sigma_x + \sigma_y\right), \quad \gamma_{zx} = 0 \ ,$$

or in matrix form

$$\{\varepsilon\} = [H]\{\sigma\}, \tag{1.32}$$

where

$$[H] = \frac{1}{E}\begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix}. \tag{1.33}$$

Deformation ε_z depends on deformations ε_x and ε_y

$$\varepsilon_z = -\frac{\mu}{1-\mu}\left(\varepsilon_x + \varepsilon_y\right). \tag{1.34}$$

Towards stresses the relations are

$$\sigma_x = \frac{E}{1-\mu^2}\left(\varepsilon_x + \mu\varepsilon_y\right),\ \tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy}$$

$$\sigma_y = \frac{E}{1-\mu^2}\left(\varepsilon_y + \mu\varepsilon_x\right), \quad \tau_{yz} = 0 \tag{1.35}$$

$$\sigma_z = 0, \quad \tau_{zx} = 0 .$$

or in matrix form as in (1.23), where

$$[E] = \begin{bmatrix} \frac{E}{1-\mu^2} & \frac{\mu E}{1-\mu^2} & 0 \\ \frac{\mu E}{1-\mu^2} & \frac{E}{1-\mu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\mu)} \end{bmatrix} . \tag{1.36}$$

Lampe's equations are also simplified and acquire the following form

$$G\nabla^2 u + \frac{1}{1-\mu}\frac{\partial\theta}{\partial x} + R_x = 0$$

$$G\nabla^2 v + \frac{1}{1-\mu}\frac{\partial\theta}{\partial y} + R_y = 0 , \tag{1.37}$$

where $\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

1.4.2. Two Dimensional Strain State

Character feature in this case is that the complete displacements of all body's particles are parallel to one plane and are function of two parameters. Such case is shown on fig. 1.5.

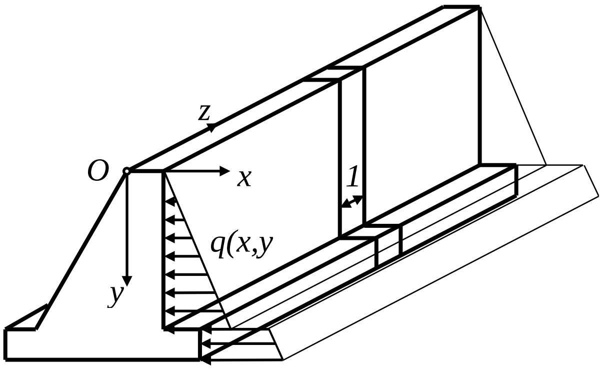

Fig. 1.5

Body's particles laying on cross-sections far enough from the both ends of the body remain in their initial plane. To investigate the stress and strain states of the body, it is enough to consider one element between cross-sections at a unity distance. As far as the displacements are only in plane parallel to the cross-section of the body, it can be written

$$u = f_1(x,y), \quad v = f_2(x,y), \quad w = 0 . \tag{1.38}$$

Under these conditions Cauchy's equations acquire the following form

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{yz} = 0, \quad \gamma_{zx} = 0, \quad \varepsilon_z = 0\,(v_z = 0) \quad . \tag{1.39}$$

According Hook's law, it is obtained

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right] = 0\,, \tag{1.40}$$

or

$$\sigma_z = \mu\left(\sigma_x + \sigma_y\right) = \mu I\,, \tag{1.41}$$

where $I = \sigma_x + \sigma_y$ is invariant of the stress tensor. The remaining relations according Hook's law are

$$\begin{aligned} &\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right], \quad \gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\mu)}{E}\tau_{xy} \\ &\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_z + \sigma_x\right)\right], \quad \gamma_{yz} = 0 \\ &\varepsilon_z = 0\,, \quad \gamma_{zx} = 0 \quad . \end{aligned} \tag{1.42}$$

In matrix form these equations are written as in (1.24), where

$$[H] = \frac{1}{E}\begin{bmatrix} \left(1-\mu^2\right) & \frac{\mu}{1-2\mu} & 0 \\ \frac{\mu}{1-2\mu} & \frac{1-\mu}{1-2\mu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{1.43}$$

The relationship stress-strain is similar to (1.23), where

$$[E] = \frac{E}{1+\mu}\begin{bmatrix} \frac{1-\mu}{1-2\mu} & \frac{\mu}{1-2} & 0 \\ \frac{\mu}{1-2} & \frac{1-\mu}{1-2\mu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{1.44}$$

Most convenient for solving 2D elasticity theory problem is to use stresses. For both types 2D problems can be written one and the same equilibrium differential equations, one and the same Morris-Levi's equation, which is obtained by compatibility strain equations, where the deformations are expressed by the stresses according Hook's law. In a similar manner are obtained the surface equations.

1.5. Axis-Symmetrical Problem

Axis-Symmetrical bodies are obtained by rotating a plane model around an axis (fig. 1.6, a).

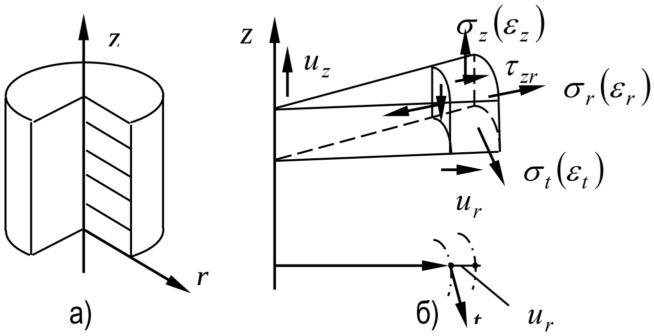

Fig. 6

From Mathematical point of view the above case is analogue to plane stress and plane strain states case. Because of the symmetry strain, and therefore the stress states in arbitrarily cross-section along the axis of symmetry, are completely defined by two displacement's components. If we denote radial and axial coordinates of the point with *r* and *z*, and with u_r and u_z respectively the displacement along *r* and *z*, then we can write about deformations the following thing (fig. 1.6, б)

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}, \quad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$
$$\varepsilon_t = \frac{2\pi(r + u_r - 2\pi r)}{2\pi r} = \frac{u_r}{r} \quad . \tag{1.45}$$

The deformation matrix about axis-symmetrical problems is

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_t \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_r}{\partial r} \\ \frac{\partial u_z}{\partial z} \\ \frac{u_r}{r} \\ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \end{Bmatrix} = [D]\{u\}, \tag{1.46}$$

where $\{u\} = \begin{Bmatrix} u_r \\ u_z \end{Bmatrix}, \quad [D] = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ 0 & \frac{1}{r} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}.$ (1.47)

The relationship stress-strain is

$$\{\sigma\} = [E]\{\varepsilon\}, \tag{1.48}$$

where $\{\sigma\}=\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_t\\ \tau_{rt}\end{Bmatrix}$, $[E]=\frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\begin{bmatrix}1 & \frac{\mu}{1-\mu} & \frac{\mu}{1-\mu} & 0\\ \frac{\mu}{1-\mu} & 1 & \frac{\mu}{1-\mu} & 0\\ \frac{\mu}{1-\mu} & \frac{\mu}{1-\mu} & 1 & 0\\ 0 & 0 & 0 & \frac{1-2\mu}{2(1-\mu)}\end{bmatrix}$. (1.49)