Stresses and deformations at torsion

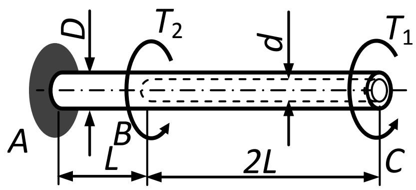
Example

A steel shaft is loaded as shown. Knowing that T_1 = 1200 N.m, T_2 = 1300 N.m, L = 1 m, d = 45 mm, D = 52 mm determine:

a) the maximum shearing stress in the shaft and check whether it can bear the loading;

b) the angle of twist φ_{AC}.

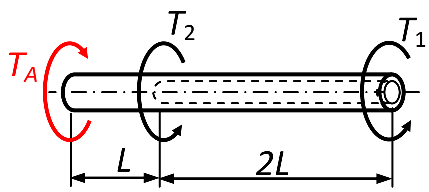
The material is standard steel S355JR with yielding stress σ_y = 355 MPa and shear modulus G = 0,8.10^{11} Pa. Use a factor of safety FS = 1,29.



Solution

1. *Free Body Diagram (FBD)*

Since this is pure torsion and all loads are torques around the axis of the shaft there will be only one support reaction which is different from zero in the point of the support - T_A



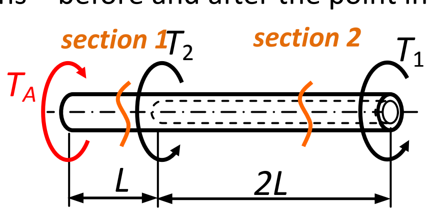
2. *Determining the support reactions by using the equilibriums*

The momentum equilibrium about the axis of the shaft gives

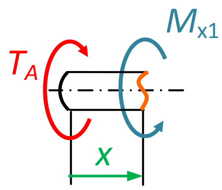
$$\sum M_x = 0 \;\Rightarrow\; T_A - T_1 - T_2 = 0 \;\Rightarrow\; T_A = T_1 + T_2 = 1200 + 1300 \;\Rightarrow\; \boxed{T_A = 2500 \text{ N.m}}$$

3. *Applying the method of the section to determine the internal forces by using the equilibriums*

There will be 2 sections – before and after the point in which T_2 is applied



For ***section 1*** the left part will be taken

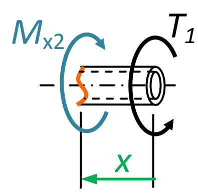


A coordinate x is used to determine the position of the section (the exact position of the section is not given). It starts from point *A* and changes to the right. The boundaries of this coordinate are $0 \le x \le L$.

The momentum equilibrium about the axis of the shaft is applied to determine the internal force M_{X1}

$$\sum M_x = 0 \;\Rightarrow\; T_A - M_{x1} = 0 \;\Rightarrow\; \mathrm{M}_{x1} = T_A \;\Rightarrow\; \boxed{M_{x1} = 2500 \text{ N.m}}$$

For ***section 2*** the right part will be taken



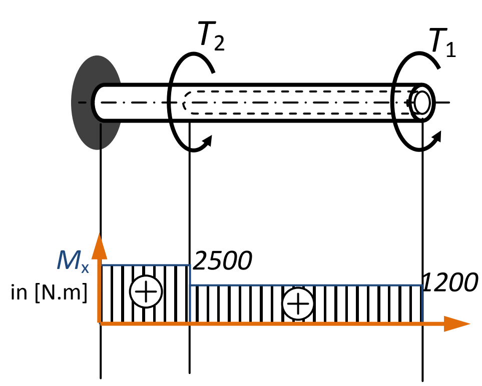
A coordinate x is again used to determine the position of the section but for this section it starts from point *C* and changes to the left and its boundaries are $0 \le x \le 2L$.

The momentum equilibrium about the axis of the shaft is applied to determine the internal force M_{X2}

$$\sum M_x = 0 \;\Rightarrow\; T_1 - M_{x1} = 0 \;\Rightarrow\; \mathrm{M}_{x1} = T_1 \;\Rightarrow\; \boxed{M_{x1} = 1200 \text{ N.m}}$$

4. *Internal forces diagrams*

In both sections the internal force M_x is constant so its diagram looks like:



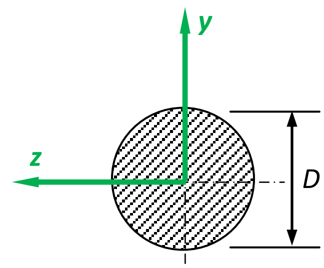
5. Obtaining the geometrical properties of the construction cross-section

The cross-section of the structure consists of 2 parts.

First part – solid circle

Using the formulas for the polar moment of inertia of a solid circle (*see Strength of Materials Handbook, page 6*) it can be written

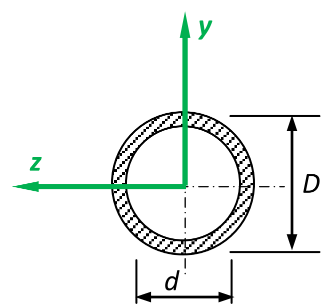
$$I_{C1} = \frac{\pi D^4}{32} = \frac{3.14(52)^4}{32} = 717452{,}3\text{mm}^4$$



Second part – pipe

Using the formulas for the polar moment of inertia of a pipe (*see Strength of Materials Handbook, page 6*) it can be written

$$I_{C2} = \frac{\pi\left(D^4 - d^4\right)}{32} = \frac{3.14\left(52^4 - 45^4\right)}{32} = 315078{,}5\text{mm}^4$$



6. Applying the formulas for finding the stresses

The formula for the shearing stresses in torsion τ_t is

$$\tau_t = \frac{M_x}{I_C}.\rho$$

where ρ is a polar coordinate in the cross-section which origin is in the centrioid. The maximum stresses will be calculated when ρ is maximum that is for the external contour of the cross-section.

Stress in the first part of the shaft - solid circle

$$\max \tau_{t1} = \frac{M_{x1}}{I_{C1}}.\max \rho_1$$

The maximum value of the coordinate of the external contour of the cross-section for the first part is

$$\max \rho_1 = D = 52\text{mm}$$

Stress is a unit which dimension is Pa = N/m^2. That is why in the formula for τ_t all units must be in N and m (*see Strength of Materials Handbook, page 39 if you cannot convert units*). That means:

$$M_{x1} = 2500\text{N.m}$$

$$I_{C1} = 717452{,}3\text{mm}^4 = 717452{,}3.10^{-12}\,\text{m}^4 = 0{,}7175.10^{-6}\,\text{m}^4$$

$$\max \rho_1 = 52\text{mm} = 52.10^{-3}\text{m}$$

Now the shearing stresses in the first part of the shaft can be calculated

$$\max \tau_{t1} = \frac{M_{x1}}{I_{C1}}.\max \rho_1 = \frac{2500}{0{,}7175.10^{-6}}.52.10^{-3} = 181184{,}7.10^3 = 181{,}2.10^6\,\text{Pa} = 181{,}2\text{MPa}$$

Stress in the second part of the shaft - pipe

$$\max \tau_{t2} = \frac{M_{x2}}{I_{C2}}.\max \rho_2$$

The maximum value of the coordinate of the external contour of the cross-section for the second part is

$$\max \rho_2 = D = 52\text{mm} = 52.10^{-3}\text{m}$$

The other parameters are:

$$M_{x2} = 1200\text{N.m}$$

$$I_{C2} = 315078{,}5\text{mm}^4 = 315078{,}5.10^{-12}\,\text{m}^4 = 0{,}3151.10^{-6}\,\text{m}^4$$

Now the shearing stresses in the second part of the shaft can be calculated

$$\max \tau_{t2} = \frac{M_{x2}}{I_{C2}}.\max \rho_2 = \frac{1200}{0{,}3151.10^{-6}}.52.10^{-3} = 198032{,}4.10^3 = 198.10^6\,\text{Pa} = 198\text{MPa}$$

7. *Estimation of the construction*

The results show that the maximum stress is in the second part of the shaft

$$\max \tau_t = \max \tau_{t2} = 198\text{MPa}$$

Since the material of the structure is standard steel S355JR with yielding stress σ_y = 355 MPa (*see Strength of Materials Handbook, page 34*) and the factor of safety is FS = 1,29 the allowable stresses σ_{all} can be calculated.

$$\sigma_{all} = \frac{\sigma_y}{FS} = \frac{355}{1,29} = 275,2\text{MPa}$$

These are normal stresses. The allowable shearing stresses can be determined by accepting that

$$\tau_{all} = 0,5.\sigma_{all} = 0,5.275,2 = 137,6\text{MPa}$$

The allowable stresses are used as a criteria for estimation of the construction.

If the following condition is kept

$$\left|\max \tau_t\right| \leq \tau_{all}$$

the construction is suitable for the applied loading.

Since for this shaft

$$\left|\max \tau_t\right| = 198\text{MPa} \geq \tau_{all}$$

it can be concluded that ***the shaft is not appropriate for the applied loads and bigger cross-section has to be used.***

8. *Calculation of the angle of twist φ_{AC}*

The formula for calculation of the angle of twist for constant internal force is:

$$\varphi_{AC} = \frac{M_x}{G.I_C}.L_c$$

where L_c is the current length of the investigated part of the shaft.

Since the shaft consists of two parts it follows that:

$$\varphi_{AC} = \varphi_{AB} + \varphi_{BC}$$

Angle of twis for the **first part**:

The current length of the first part is $L_{c1} = L = 1$m

$$\varphi_{AB} = \frac{M_{x1}}{G.I_{C1}}.L_{c1} = \frac{2500}{0,8.10^{11}.0,7175.10^{-6}}.1 = 4355,4.10^{-5} = 0,043554\text{rad}$$

Angle of twis for the **second part**:

The current length of the second part is $L_{c2} = 2L = 2\text{m}$

$$\varphi_{BC} = \frac{M_{x2}}{G.I_{c2}}.L_{c2} = \frac{1200}{0,8.10^{11}.0,3151.10^{-6}}.2 = 9520,8.10^{-5} = 0,095208\text{rad}$$

The angle of twist for the whole shaft is

$$\varphi_{AC} = \varphi_{AB} + \varphi_{BC} = 0,043554 + 0,095208 = 0,138762\text{rad}$$