CENTROID

The following planar shape can be regarded (Fig.1). It consists of 3 shapes. For each shape the position of the centroid and the area are known. The areas are assigned with A_1, A_2, A_3 and their centroids are C_1, C_2, C_3 respectively. For the particular example a coordinate system y-z will be used. It will be demonstrated how to find the vertical coordinate of the common centroid $\boldsymbol{C}$. The vertical coordinates of the separate shapes are y_{C1}, y_{C2} and y_{C3} (Fig.1). The following formula is used for determining the coordinate $\boldsymbol{y_C}$:

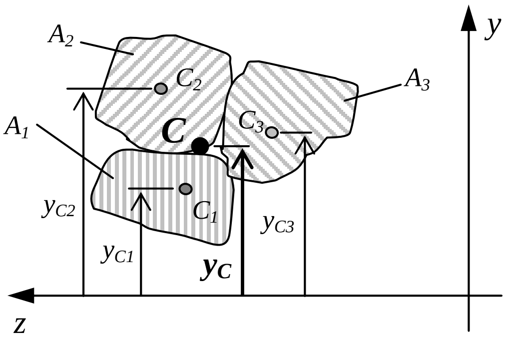


Fig. 1

$$y_C = \frac{y_{C1}.A_1 + y_{C2}.A_2 + y_{C3}.A_3}{A_1 + A_2 + A_3} = \frac{\sum_{i=1}^{3} y_{Ci}.A_i}{\sum_{i=1}^{3} A_i}.$$

If the figure consists of n shapes it can be written:

$$y_C = \frac{\sum_{i=1}^{n} y_{Ci}.A_i}{\sum_{i=1}^{n} A_i}.$$

Finding the horizontal coordinate of the common centroid is analogical and the general formula for a figure consisting of n shapes is:

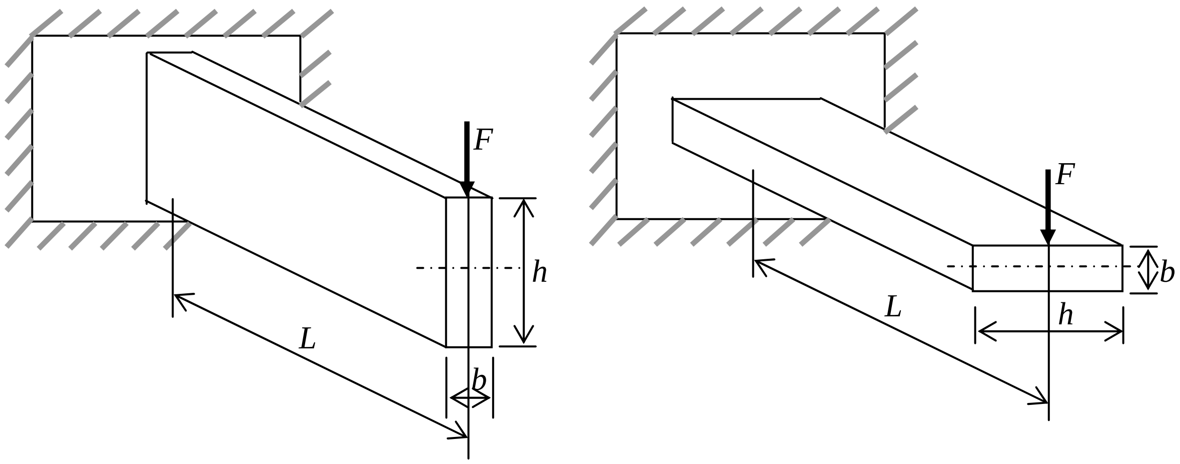
$$z_C = \frac{\sum_{i=1}^{n} z_{Ci}.A_i}{\sum_{i=1}^{n} A_i}.$$

The following rules have to be minded when the position of the centroid has to be determined:

- if the figure has one axis of symmetry the common centroid is lying on this axis;
- if the figure has two or more axis of symmetry (as for instance a rectangle or a circle) the position of the common centroid is in the cross point of these axis.

MOMENTS OF INERTIA

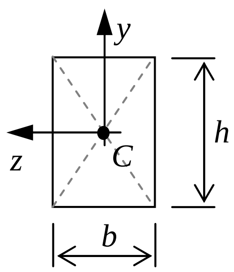
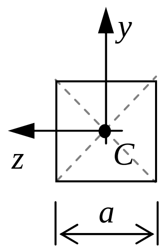
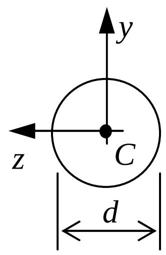
Moments of inertia account how an area is situated with respect to an axis. Their physical meaning is to represent how a construction can resist bending. The bigger the moment of inertia of the cross-section is the bigger are the transverse loads that can be applied to a construction. As for instance the beam shown on Fig.2 (a) can bare bigger force *F* than the one from Fig.2 (b) though the area of the cross-section is one and the same ($A=h.b$).



(a) **(b)**

Fig. 2

That can be explained by the fact that the cross-section of the first construction has bigger moment of inertia according to the horizontal axis (the bending is around this axis) which is obvious from the formulas for the moment of inertia of a rectangle given in the table below – $I_z>I_y$ because $h>b$.

Moments of inertia with respect to the axis passing through the centroid for some simple shapes		
Rectangle	**Square**	**Circle**
		
$I_y=\frac{h.b^3}{12}$; $I_z=\frac{h^3.b}{12}$	$I_y=I_z=\frac{a^4}{12}$	$I_y=I_z=\frac{\pi d^4}{64}$

If the moment of inertia with respect to an axis which is not passing through the centroid has to be determined, the following formulas are used:

$$I_{y*} = I_y + e^2.A\,,$$

$$I_{z*} = I_z + f^2.A\,,$$

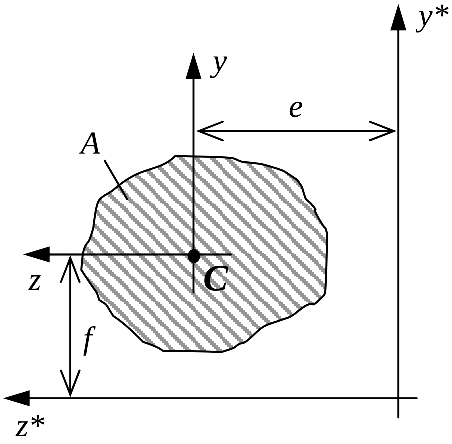


Fig. 3

where A is the area of the shape, e is the distance between axis y and $y*$, f is the one between z and $z*$.

This is the theorem of Steiner. The terms which are added to the moments of inertia are called Steiner's additions.

As for instance for a rectangular shape (Fig.4) can be written

$$I_{y*} = I_y + e^2 A = \frac{h.b^3}{12} + e^2(h.b)\,,$$

$$I_{z*} = I_z + f^2 A = \frac{h^3.b}{12} + f^2(h.b)\,.$$

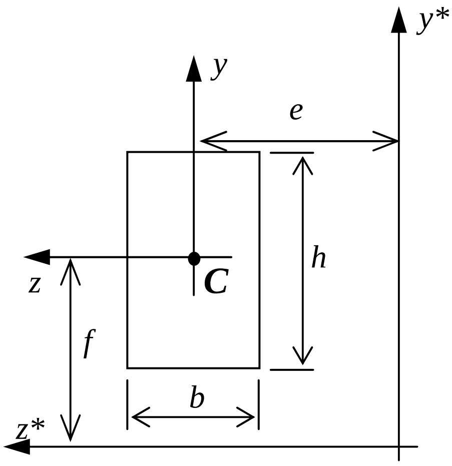


Fig. 4

EXAMPLE

For the shape shown on Fig.5 determine the position of the centroid and the principle moments of inertia.

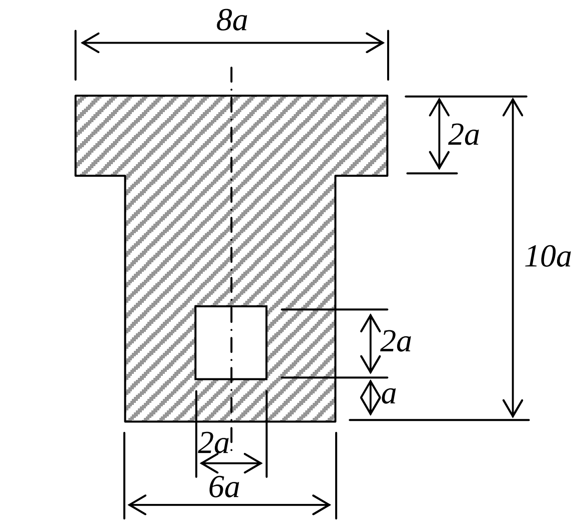


Fig. 5

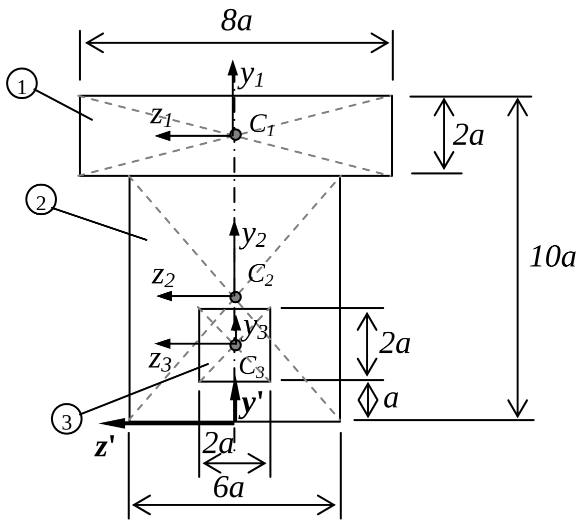


Fig. 6

Solution: The shape has one axis of symmetry that means that the centroid is lying on it.

For the purpose of finding the centroid the shape has to be represented as a sum of simple shapes (Fig.6) – in the particular example two rectangles and one square. Later it has to be minded that because the square is hollow it has to be subtracted when determining the position of the centroid and the moment of inertia.

The centroid of each shape C_i (i = 1,2,3) is used as an origin of a local coordinate system (Fig.6).

The position of the common centroid has to be determined with respect to some auxiliary coordinate system. This coordinate system can be placed anywhere but for convenience it is recommended that if the shape has an axis of symmetry one of the axis of the coordinate system has to coincide with it. In this example this will be the vertical axis y'. The horizontal one z' will coincide with the lowest line of the shape. Now the position of the centroid will be found according to the auxiliary system $y'z'$ (Fig.6).

Only the vertical coordinate of the centroid will be searched because of the axis of symmetry (the centroid is lying on it).

The following formula is used:

$$y'_C = \frac{y'_{C1}.A_1 + y'_{C2}.A_2 - y'_{C3}.A_3}{A_1 + A_2 - A_3}.$$

There is minus in front of A_3 because the square is hollow.

The areas of the separate shapes are

$$A_1=2a.8a,$$
$$A_2=8a.6a,$$
$$A_3=2a.2a.$$

The vertical coordinates of the centroids C_i (i=1,2,3) with respect to $y'z'$ auxiliary coordinate system are

$$y'_{C1}=9a,$$
$$y'_{C2}=4a,$$
$$y'_{C3}=2a.$$

Substitution in the formula for y'_C will lead to

$$y'_C = \frac{9a.(2a.8a)+4a.(8a.6a)-2a.(2a.2a)}{2a.8a+8a.6a-2a.2a} = \frac{144a^3+192a^3-8a^3}{16a^2+48a^2-4a^2} = \frac{328a^3}{60a^2} = 5,47a\,.$$

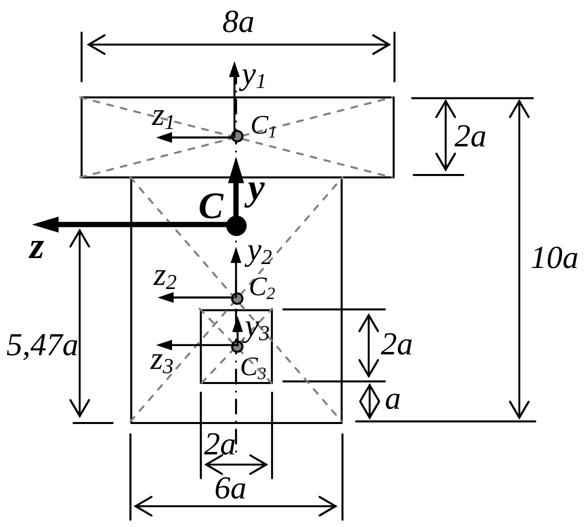


Fig. 7

Now when the position of the centroid is known one can determine the principle moments of inertia with respect to *y* and *z* axis (Fig.7).

The following can be written for the moment of inertia I_y:

$$I_y = I_{y1} + I_{y2} - I_{y3}\,.$$

Because the axis y, y_1, y_2 and y_3 are coinciding (they are all lying on the axis of symmetry) the Steiner's additions are equal to zero.

The moments of inertia of the separate shapes with respect to y_i axis (i=1,2,3) are

$$I_{y1} = \frac{(8a)^3.2a}{12},\; I_{y2} = \frac{(6a)^3.8a}{12},\; I_{y3} = \frac{(2a)^4}{12}.$$

Substituting in the expression for I_y will lead to

$$I_y = \frac{(8a)^3.2a}{12} + \frac{(6a)^3.8a}{12} - \frac{(2a)^4}{12} = \frac{1}{12}\left(1024a^4 + 1728a^4 - 16a^4\right) = \frac{2736a^4}{12} = 228a^4\,.$$

Before calculating the moment of inertia I_z it has to be taken into consideration that the axis z, z_1, z_2 and z_3 are not coinciding. The distance between these axis are as follows:

$$\overline{zz_1} = 3,53a\,,$$

$$\overline{zz_2} = 1,47a\,,$$

$$\overline{zz_3} = 3,47a\,.$$

The moments of inertia of the separate shapes with respect to z_i axis (i=1,2,3) are

$$I_{z1} = \frac{(2a)^3.8a}{12}, \qquad I_{z2} = \frac{(8a)^3.6a}{12}, \qquad I_{z3} = \frac{(2a)^4}{12}\,.$$

Using this and knowing the areas of the separate shapes which were calculated earlier the expression for I_z acquires the view

$$I_z = \left(I_{z1} + \left(\overline{zz_1}\right)^2.A_1\right) + \left(I_{z2} + \left(\overline{zz_2}\right)^2.A_2\right) - \left(I_{z3} + \left(\overline{zz_3}\right)^2.A_3\right)$$

$$I_z = \left(\frac{(2a)^3.8a}{12} + (3,53a)^2(2a.8a)\right) + \left(\frac{(8a)^3.6a}{12} + (1,47a)^2(8a.6a)\right) - \left(\frac{(2a)^4}{12} + (3,47a)^2(2a.2a)\right)$$

$$= \left(5,3a^4 + 199,4a^4\right) + \left(256a^4 + 103,7a^4\right) - \left(1,3a^4 + 48,2a^4\right) = 204,7a^4 + 359,7a^4 - 49,5a^4 = 514.9a^4$$