How to find slope in beam structures

Only 2D beam structures will be regarded blow.

Slope $\theta$ is the angle between the horizontal axis and a line which is tangent to a point from the deformed structure (see the picture below). The formula and the procedure which are described below allow obtaining the slope in a given point of the structure.

![Diagram](https://via.placeholder.com/150)

The tangent line to a point from the deformed axis of the structure

The deformed axis of the structure

Finding slope is a task which contains the following steps:

- drawing the Frere Body Diagram (FBD)
- determining the support reactions by using the equilibriums
- applying the method of the section to determine the internal forces by using the equilibriums (only $M_z$ will be determined)
- determining the derivatives of the $M_z$
- applying the formula for finding the slope

The formula with which the slope in a given point is found is very similar to the formula for the vertical deflection. The difference is in the derivative of $M_z$:

$$\theta_i = \int_{x} \frac{M_z}{E I_z} \frac{\partial M_z}{\partial M_i} \, dx$$

The result from this formula is radians or degrees.

In the above formula:

$\theta_i$ - slope in the $i$-th point,

$M_z$ – internal moment,

$E$ – modulus of elasticity (Young’s modulus),

$I_z$ – moment of inertia of the cross-section,

$\frac{\partial M_z}{\partial M_i}$ - derivative of $M_z$ with respect to the concentrated moment in the $i$-th point.

The integration is along the length of the beam.

In order to find the slope in a given point there must be a concentrated moment in this point.

**IT IS IMPORTANT TO MAKE A LETTER SOLUTION (not to replace the parameters with their values) UNTIL SUBSTITUTING IN THE FORMULA!!!**

To demonstrate the above described procedure two examples will be regarded:
**Example 1**

Find the slope in point $B (\theta_y = ?)$ if $w = 10 \text{ kN/m}$, $M = 5 \text{ kN.m}$, $L = 2 \text{ m}$. The cross-section is a standard rectangular hollow section 150x100x10 according to EN 10210-2:2006. The material is ductile standard steel S235JR with modulus of elasticity $E = 2.10^{11} \text{ Pa}$.

1. **FBD**

![Diagram of the FBD](image)

2. **Determining the support reactions by using the equilibriums**

\[ \sum F_x = 0 \Rightarrow A_x = 0 \text{ kN}; \]

\[ \sum F_y = 0 \Rightarrow A_y + B_y - Q = 0 \Rightarrow A_y + B_y = Q \]

\[ \sum M_{z_x} = 0 \Rightarrow M - Q \cdot 2L + B_y \cdot L = 0 \Rightarrow B_y = \frac{w \cdot 2L \cdot 2L}{L} - \frac{M}{L} \Rightarrow B_y = 4wL - \frac{M}{L} \]

From the second equilibrium

\[ A_y = Q - B_y \Rightarrow A_y = w \cdot 2L - \left(4wL - \frac{M}{L}\right) = \Rightarrow A_y = -2wL + \frac{M}{L} \]

3. **Applying the method of the section to determine the internal forces by using the equilibriums**

Two sections have to be made in the regarded example.

![Diagram of the two sections](image)
The left part will be taken for section 1. The internal forces are drawn in the point of the section - $S_1$.

**section 1**

$$\begin{align*}
A_x & \quad A_y \\
S_1 & \quad M_{z_1} \\
N_{x_1} & \quad \theta_B \\
V_{y_1} &
\end{align*}$$

The boundaries of the coordinate $x$ are: $0 \leq x \leq L$.

Since only $M_z$ will be used in the formula this will be the only internal force to be determined.

$$\sum M_{z_1} = 0 \Rightarrow M_{z_1} - A_y.x = 0 \Rightarrow M_{z_1} = A_y.x \Rightarrow M_{z_1} = \left(-2wl + \frac{M}{L}\right)x \Rightarrow M_{z_1} = -2wlx + \frac{M}{L}x$$

The right part will be taken for section 2. The internal forces are drawn in the point of the section - $S_2$.

**section 2**

$$\begin{align*}
Q' = w.x \\
S_2 & \quad N_{x_2} \\
V_{y_2} & \quad x \\
C &
\end{align*}$$

The boundaries of the coordinate $x$ are: $0 \leq x \leq 2L$.

Again only $M_z$ will be determined.

$$\sum M_{z_2} = 0 \Rightarrow M_{z_2} + Q'. \frac{x}{2} = 0 \Rightarrow M_{z_2} = -w.x. \frac{x}{2} \Rightarrow M_{z_2} = -\frac{wx^2}{2}.$$ 

4. **Determining the derivatives of $M_z$**

The formula for $\theta_B$ requires the derivative of $M_{z_1}$ and $M_{z_2}$ with respect to the moment $M$ which is concentrated in point $B$ (the point where $\theta_B$ is searched) to be determined. *(To find this it has to be accounted with what $M$ is multiplied in the expressions of $M_{z_1}$ and $M_{z_2}$)*

The equation of $M_{z_1}$ is

$$M_{z_1} = -2wlx + \frac{M}{L}x$$

The derivative of $M_{z_1}$ with respect to the concentrated in point $B$ moment is:
\[
\frac{\partial M_{z1}}{\partial M} = \frac{\partial M_{z2}}{\partial M} = \frac{1}{L} \frac{X}{x}
\]

The equation of \( M_{z2} \) is

\[
M_{z2} = -\frac{wx^2}{2}
\]

Since the moment \( M \) does not participate in the equation of \( M_{z2} \) the derivative will be zero

\[
\frac{\partial M_{z2}}{\partial M} = \frac{\partial M_{z2}}{\partial M} = 0
\]

5. Applying the formula for finding the slope

Since the structure consists of 2 sections 2 integrals will be used in the formula for \( \theta_b \) – one for the first section and one for the second one:

\[
\theta_b = \int_0^1 \frac{M_{z1}}{EJ_z} \frac{\partial M_{z1}}{\partial M} dx + \int_0^2 \frac{M_{z2}}{EJ_z} \frac{\partial M_{z2}}{\partial M} dx
\]

The boundaries of each integral coincide with the boundaries of each section.

Substituting the expressions for \( M_{z1} \) and \( M_{z2} \) and their derivatives with respect to \( M \) will give

\[
\theta_b = \int_0^1 \left( -\frac{2wLx}{EJ_z} + \frac{M}{L} \right) \frac{1}{L} \frac{X}{x} dx + \int_0^2 \left( -\frac{wx^2}{2EJ_z} \right) \frac{1}{L} dx
\]

The second integral will be zero since the derivative of \( M_{z2} \) with respect to \( M \) is zero.

Opening the parenthesis will lead to

\[
\theta_b = \int_0^1 \left( \frac{-2wx^2 + \frac{M}{L} x^2}{EJ_z} \right) dx = \int_0^1 \frac{(-2wx^2)}{EJ_z} dx + \int_0^1 \frac{\left( \frac{M}{L} x^2 \right)}{EJ_z} dx
\]

Moving the constants out of the integrals it will follow

\[
\theta_b = \frac{-2w}{EJ_z} \int_0^1 x^2 dx + \frac{M}{L^2 EJ_z} \int_0^1 x^3 dx = \frac{-2w \frac{L^3}{3}}{EJ_z} + \frac{M \frac{L^3}{3}}{EJ_z}
\]

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be in \([N]\), \([m]\), \([Pa]\).
The moment of inertia of the cross-section $I_z$ can be taken from the tables. For the standard shape 150x100x10 according to EN 10210-2:2006 $I_z = 1282 \text{ cm}^4 = 1282 \times 10^{-8} \text{ m}^4$.

The loads are $w = 10 \text{ kN/m} = 10 \times 10^3 \text{ N/m}$, $M = 5 \text{ kN.m} = 5 \times 10^3 \text{ N.m}$.

$$\theta_b = \frac{-2 \times 10^3}{2.10^{11} \times 1282.10^{-8}} \frac{2^3}{3} + \frac{5 \times 10^3}{2^2 2.10^{11} \times 1282.10^{-8}} \frac{2^3}{3}$$

$$\theta_b = -0.0208 + 0.0013$$

$$\theta_b = -0.0195 \text{rad}$$

The slope is measured in radians.

The above example shows how to find the slope $\theta$ in a point in which there is a concentrated moment.

The next example is related to cases in which the slope $\theta$ has to be determined in a point in which there is no concentrated moment.

**Example 2**

Find the slope in point $A$ ($\theta_A = ?$) if $w = 20 \text{ kN/m}$, $L = 2 \text{ m}$. The cross-section is a standard IPE shape – IPE 180. The material is ductile standard steel S235JR with modulus of elasticity $E = 2 \times 10^{11} \text{ Pa}$.

1. **FBD**

In this example there is no concentrated moment in point $A$ which makes it impossible to find the derivative of $M_z$. In such cases a fictitious concentrated moment $M_f$ which value is equal to zero is added in the point where $\theta$ needs to be found. The solution continues according to the steps described in the previous example.
2. Determining the support reactions by using the equilibriums

\[ \sum F_x = 0 \Rightarrow B_x = 0; \]
\[ \sum F_y = 0 \Rightarrow B_y - Q = 0 \Rightarrow B_y = wL. \]
\[ \sum M_C = 0 \Rightarrow M_B - M_f + Q \frac{L}{2} = 0 \Rightarrow M_C = M_f - wL \frac{L}{2}. \]

3. Applying the method of the section to determine the internal forces by using the equilibriums

One section is needed to determine the internal forces.

The left part will be taken for the section. The internal forces are drawn in the point of the section - \( S_1 \).

The boundaries of the coordinate \( x \) are: \( 0 \leq x \leq L \).

Since only \( M_z \) will be used in the formula this will be the only internal force to be determined.

\[ \sum M_{z_1} = 0 \Rightarrow M_{z_1} - M_f + Q \frac{x}{2} = 0 \Rightarrow M_{z_1} = M_f - wL \frac{x^2}{2} \Rightarrow M_{z_1} = M_f - wL \frac{x^2}{2}. \]
4. **Determining the derivatives of** $M_z$

Now the derivative of $M_{z1}$ with respect to the moment $M_f$ which is concentrated in point $A$ (the point where $\theta_A$ is searched) can be determined. *(To find this it has to be accounted with what $M_f$ is multiplied in the expression of $M_{z1}$)*

\[ \frac{\partial M_{z1}}{\partial M_A} = \frac{\partial M_{z1}}{\partial M_f} = 1 \]

5. **Applying the formula for finding the deflection**

The formula for $\theta_A$ consists of one integral which boundaries are the same as the boundaries of the section:

\[ \theta_A = \int_0^L \frac{M_{z1}}{E I_z} \cdot \frac{\partial M_{z1}}{\partial M_A} \, dx \]

Substituting the expression for $M_{z1}$ and its derivative with respect to $M_f$ will give:

\[ \theta_A = \int_0^L \left( \frac{M_f - w \frac{x^2}{2}}{E I_z} \right) \, dx \]

This integral can be simplified when accounting that $M_f$ is zero:

\[ \theta_A = \int_0^L \left( \frac{-w \frac{x^2}{2}}{E I_z} \right) \, dx \]

Opening the parenthesis and moving the constants out of the integral will lead to:

\[ \theta_A = -\frac{w}{2 E I_z} \int_0^L x^2 \, dx = -\frac{w L^3}{2 E I_z} \left( \frac{2}{3} \right) \]

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be [N], [m], [Pa].

The moment of inertia of the cross-section $I_z$ can be taken from the tables. For the standard IPE shape IPE 180 $I_z = 1317 \, \text{cm}^4 = 1317 \times 10^{-8} \, \text{m}^4$.

The load is $w = 20 \, \text{kN/m} = 20 \times 10^3 \, \text{N/m}$.

\[ \theta_A = \frac{20 \times 10^3}{2.2 \times 10^{11} \times 1317 \times 10^{-8}} \left( \frac{2^3}{3} \right) \quad \Rightarrow \quad \theta_A = -0.0101 \, \text{rad} \]