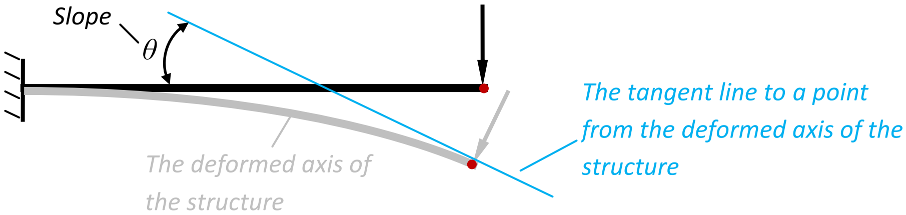
How to find slope in beam structures

Only 2D beam structures will be regarded blow.

Slope θ is the angle between the horizontal axis and a line which is tangent to a point from the deformed structure (see the picture below). The formula and the procedure which are described below allow obtaining the slope in a given point of the structure.



Finding slope is a task which contains the following steps:

- drawing the Frere Body Diagram (FBD)
- determining the support reactions by using the equilibriums
- applying the method of the section to determine the internal forces by using the equilibriums (only M_z will be determined)
- determining the derivatives of the M_z
- applying the formula for finding the slope

The formula with which the slope in a given point is found is very similar to the formula for the vertical deflection. The difference is in the derivative of M_z:

$$\theta_i = \int_L \frac{M_z}{E.I_z}.\frac{\partial M_z}{\partial M_i}.dx$$

The result from this formula is radians or degrees.

In the above formula:

θ_i - slope in the i-th point,

M_z – internal moment,

E – modulus of elasticity (Young's modulus),

I_z – moment of inertia of the cross-section,

$\frac{\partial M_z}{\partial M_i}$ - derivative of M_z with respect to the concentrated moment in the *i*-th point.

The integration is along the length of the beam.

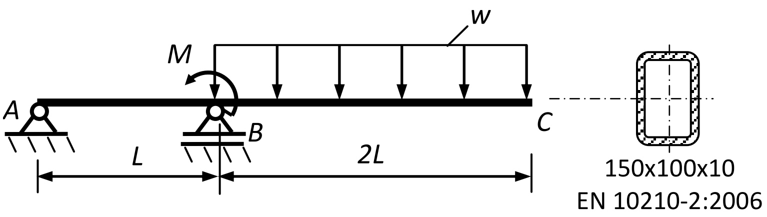
In order to find the ***slope*** in a given point there must be a ***concentrated moment*** in this point.

<u>IT IS IMPORTANT TO MAKE A LETTER SOLUTION (not to replace the parameters with their values) UNTIL SUBSTITUTING IN THE FORMULA!!!</u>

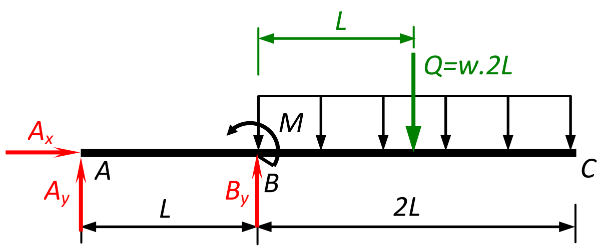
To demonstrate the above described procedure two examples will be regarded:

Example 1

Find the slope in point *B* ($\theta_B = ?$) if *w* = 10 kN/m, *M* = 5 kN.m, *L* = 2 m. The cross-section is a standard rectangular hollow section 150x100x10 according to EN 10210-2:2006. The material is ductile standard steel S235JR with modulus of elasticity $E = 2.10^{11}$ Pa.



1. *FBD*



2. *Determining the support reactions by using the equilibriums*

$$\sum F_x = 0 \;\Rightarrow\; \boxed{A_x = 0 \text{ kN}};$$

$$\sum F_y = 0 \;\Rightarrow\; A_y + B_y - Q = 0 \;\Rightarrow\; A_y + B_y = Q$$

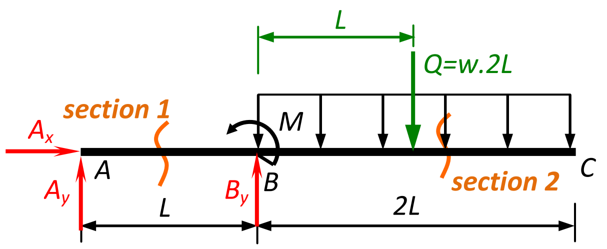
$$\sum M_{z_A} = 0 \;\Rightarrow\; M - Q.2L + B_y.L = 0 \;\Rightarrow\; B_y = \frac{w.2L.2L}{L} - \frac{M}{L} \;\Rightarrow\; \boxed{B_y = 4wL - \frac{M}{L}}$$

From the second equilibrium

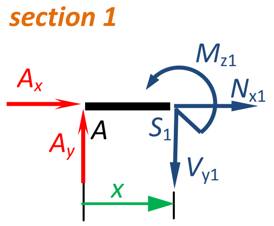
$$\Rightarrow\; A_y = Q - B_y \;\Rightarrow\; A_y = w.2L - \left(4wL - \frac{M}{L}\right) = \;\Rightarrow\; \boxed{A_y = -2wL + \frac{M}{L}}$$

3. *Applying the method of the section to determine the internal forces by using the equilibriums*

Two sections have to be made in the regarded example



The left part will be taken for ***section 1***. The internal forces are drawn in the point of the section - S_1.

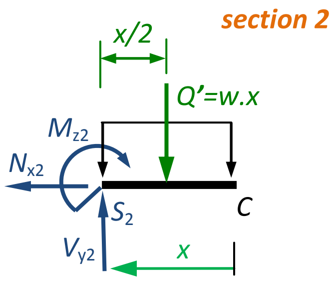


The boundaries of the coordinate x are: $0 \le x \le L$.

Since only M_z will be used in the formula this will be the only internal force to be determined.

$$\sum M_{z_{S1}} = 0 \Rightarrow M_{z1} - A_y.x = 0 \Rightarrow M_{z1} = A_y.x \Rightarrow M_{z1} = \left(-2wL + \frac{M}{L}\right)x \Rightarrow \boxed{M_{z1} = -2wLx + \frac{M}{L}x}$$

The right part will be taken for ***section 2***. The internal forces are drawn in the point of the section - S_2.



The boundaries of the coordinate x are: $0 \le x \le 2L$.

Again only M_Z will be determined.

$$\sum M_{z_{S2}} = 0 \Rightarrow M_{z2} + Q'.\frac{x}{2} = 0 \Rightarrow M_{z2} = -w.x.\frac{x}{2} \Rightarrow \boxed{M_{z2} = -\frac{wx^2}{2}}.$$

4. *Determining the derivatives of M_z*

The formula for θ_B requires the derivative of M_{z1} and M_{z2} with respect to the moment M which is concentrated in point B (the point where θ_B is searched) to be determined. (*To find this it has to be accounted with what M is multiplied in the expressions of M_{z1}* and M_{z2})

The equation of M_{z1} is

$$M_{z1} = -2wLx + \frac{M}{L}x$$

The derivative of M_{z1} with respect to the concentrated in point B moment is:

$$\boxed{\frac{\partial M_{z1}}{\partial M_B} = \frac{\partial M_{z1}}{\partial M} = \frac{1}{L}x}$$

The equation of M_{z2} is

$$M_{z2} = -\frac{wx^2}{2}$$

Since the moment M does not participate in the equation of M_{z2} the derivative will be zero

$$\boxed{\frac{\partial M_{z2}}{\partial M_B} = \frac{\partial M_{z2}}{\partial M} = 0}$$

5. *Applying the formula for finding the slope*

Since the structure consists of 2 sections 2 integrals will be used in the formula for θ_B – one for the first section and one for the second one:

$$\theta_B = \int_0^L \frac{M_{z1}}{E.I_z} \cdot \frac{\partial M_{z1}}{\partial M_B} dx + \int_0^{2L} \frac{M_{z2}}{E.I_z} \cdot \frac{\partial M_{z2}}{\partial M_B} dx$$

The boundaries of each integral coincide with the boundaries of each section.

Substituting the expressions for M_{z1} and M_{z2} and their derivatives with respect to M will give

$$\theta_B = \int_0^L \frac{\left(-2wLx + \frac{M}{L}x\right)}{E.I_z} \cdot \left(\frac{1}{L}x\right) dx + \int_0^{2L} \frac{\left(-\frac{wx^2}{2}\right)}{E.I_z} .0dx$$

The second integral will be zero since the derivative of M_{z2} with respect to M is zero.

Opening the parenthesis will lead to

$$\theta_B = \int_0^L \frac{\left(-2wx^2 + \frac{M}{L^2}x^2\right)}{E.I_z} dx = \int_0^L \frac{\left(-2wx^2\right)}{E.I_z} dx + \int_0^L \frac{\left(\frac{M}{L^2}x^2\right)}{E.I_z} dx$$

Moving the constants out of the integrals it will follow

$$\theta_B = \frac{-2w}{E.I_z} \int_0^L x^2 dx + \frac{M}{L^2 E.I_z} \int_0^L x^2 dx = \frac{-2w}{E.I_z} \frac{L^3}{3} + \frac{M}{L^2 E.I_z} \frac{L^3}{3}$$

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be in [N], [m], [Pa].

The moment of inertia of the cross-section I_z can be taken from the tables. For the standard shape 150x100x10 according to EN 10210-2:2006 I_z = 1282 cm^4 = 1282.10^{-8} m^4.

The loads are w = 10 kN/m= 10.10^3 N/m, M = 5 kN.m = 5.10^3 N.m .

$$\theta_B = \frac{-2.10.10^3}{2.10^{11}.1282.10^{-8}} \frac{2^3}{3} + \frac{5.10^3}{2^2 2.10^{11}.1282.10^{-8}} \frac{2^3}{3}$$

$$\theta_B = -0{,}0208 + 0{,}0013$$

$$\theta_B = -0{,}0195 rad$$

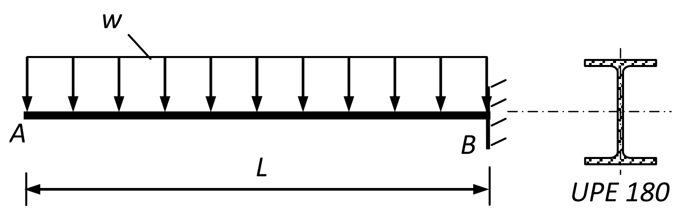
The slope is measured in radians.

The above example shows how to find the slope θ in a point in which there is a concentrated moment.

The next example is related to cases in which the slope θ has to be determined in a point in which there is no concentrated moment.

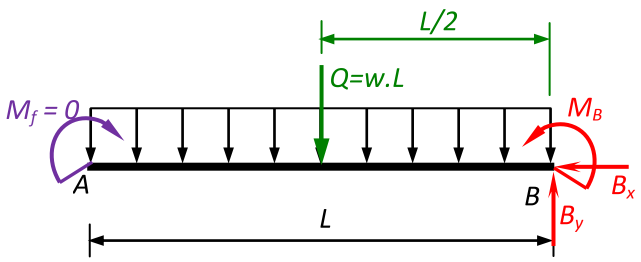
Example 2

Find the slope in point A ($\theta_A = ?$) if w = 20 kN/m, L = 2 m. The cross-section is a standard IPE shape – IPE 180. The material is ductile standard steel S235JR with modulus of elasticity E = 2.10^{11} Pa.



1. *FBD*

In this example there is no concentrated moment in point A which makes it impossible to find the derivative of M_z. In such cases a fictitious concentrated moment M_f which value is equal to zero is added in the point where θ needs to be found. The solution continues according to the steps described in the previous example.



2. *Determining the support reactions by using the equilibriums*

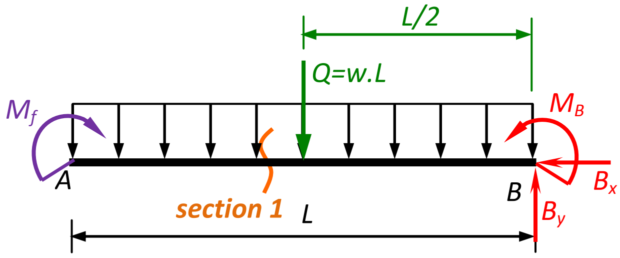
$$\sum F_x = 0 \Rightarrow \boxed{B_x = 0};$$

$$\sum F_y = 0 \Rightarrow B_y - Q = 0 \Rightarrow \boxed{B_y = w.L}.$$

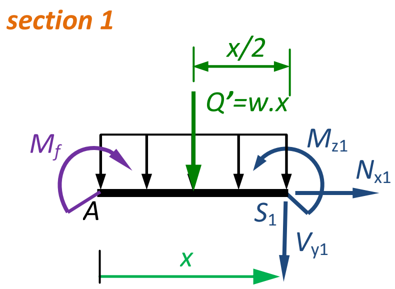
$$\sum M_{z_C} = 0 \Rightarrow M_B - M_f + Q.\frac{L}{2} = 0 \Rightarrow \boxed{M_C = M_f - w.L.\frac{L}{2}}$$

3. *Applying the method of the section to determine the internal forces by using the equilibriums*

One section is needed to determine the internal forces.



The left part will be taken for the section. The internal forces are drawn in the point of the section - S_1.



The boundaries of the coordinate x are: $0 \le x \le L$.

Since only M_z will be used in the formula this will be the only internal force to be determined.

$$\sum M_{z_{S1}} = 0 \Rightarrow M_{z1} - M_f + Q'.\frac{x}{2} = 0 \Rightarrow M_{z1} = M_f - w.x.\frac{x}{2} \Rightarrow \boxed{M_{z1} = M_f - w.\frac{x^2}{2}}$$

4. *Determining the derivatives of M_z*

Now the derivative of M_{z1} with respect to the moment M_f which is concentrated in point *A* (the point where θ_A is searched) can be determined. (*To find this it has to be accounted with what M_f is multiplied in the expression of M_{z1}*)

$$\boxed{\frac{\partial M_{z1}}{\partial M_A} = \frac{\partial M_{z1}}{\partial M_f} = 1}$$

5. *Applying the formula for finding the deflection*

The formula for θ_A consists of one integral which boundaries are the same as the boundaries of the section:

$$\theta_A = \int_0^L \frac{M_{z1}}{E.I_z}.\frac{\partial M_{z1}}{\partial M_A}dx$$

Substituting the expression for M_{z1} and its derivative with respect to M_f will give:

$$\theta_A = \int_0^L \frac{\left(M_f - w.\frac{x^2}{2}\right)}{E.I_z}.1dx$$

This integral can be simplified when accounting that M_f is zero:

$$\theta_A = \int_0^L \frac{\left(-w.\frac{x^2}{2}\right)}{E.I_z}.1dx$$

Opening the parenthesis and moving the constants out of the integral will lead to:

$$\theta_A = -\frac{w}{2E.I_z}\int_0^L x^2 dx = -\frac{w}{2E.I_z}\frac{L^3}{3}$$

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be [N], [m], [Pa].

The moment of inertia of the cross-section I_z can be taken from the tables. For the standard IPE shape IPE 180 I_z = 1317 cm^4 = 1317.10^{-8} m^4.

The load is w = 20 kN/m = 20.10^3 N/m.

$$\theta_A = \frac{20.10^3}{2.2.10^{11}.1317.10^{-8}}\frac{2^3}{3} \quad \Rightarrow \quad \boxed{\theta_A = -0{,}0101\ rad}$$