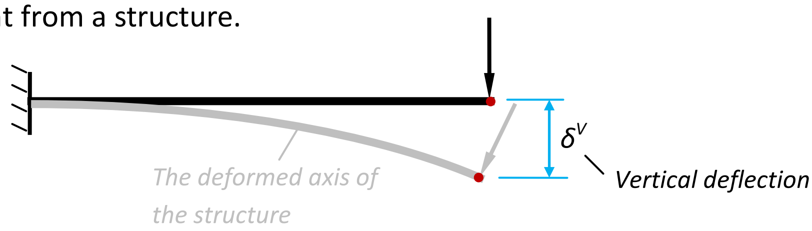
# How to find vertical deflection in beam structures

Only 2D beam structures will be regarded blow. Only vertical deflection will be discussed.

Vertical deflection is the vertical distance between a point from the undeformed axis of a structure and the same point which lies on the deformed axis. (see the picture below). The formula and the procedure which are described below allow obtaining the vertical deflection in a given point from a structure.




Finding vertical deflection is a task which contains the following steps:

- drawing the Frere Body Diagram (FBD)
- determining the support reactions by using the equilibriums
- applying the method of the section to determine the internal forces by using the equilibriums (only $M_z$ will be determined)
- determining the derivatives of the $M_z$
- applying the formulas for finding deflections

The following formula can be used for finding the vertical deflection in 2D loading in bending:

$$\delta_i^V = \int_L \frac{M_z}{E.I_z} . \frac{\partial M_z}{\partial F_i} .dx$$

In the above formula:

$\delta_i^V$ - vertical deflection in the i-th point,

$M_z$ – internal moment,

$E$ – modulus of elasticity (Young's modulus),

$I_z$ – moment of inertia of the cross-section,

$\frac{\partial M_z}{\partial F_i}$ - derivative of $M_z$ with respect to the concentrated force in the *i*-th point.

The integration is along the length of the beam.

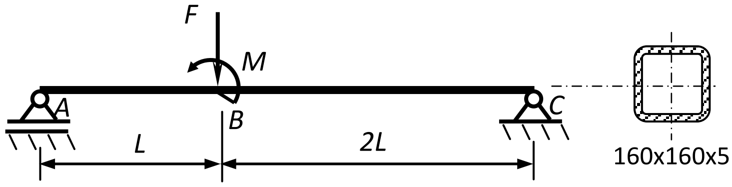
In order to find the ***vertical deflection*** in a given point there must be a ***concentrated force*** in this point.

**<u>IT IS IMPORTANT TO MAKE A LETTER SOLUTION (not to replace the parameters with their values) UNTIL SUBSTITUTING IN THE FORMULA!!!</u>**

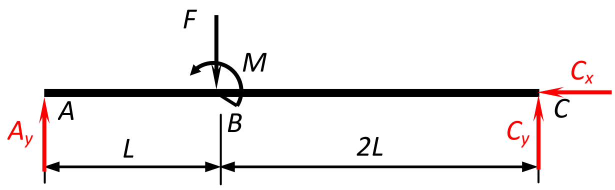
To demonstrate the above described procedure two examples will be regarded:

***Example 1***

Find the vertical deflection in point $B$ ($\delta_B^V = ?$) if $F$ = 18 kN, $M$ = 6 kN.m, $L$ = 3 m. The cross-section is a standard square hollow section 160x160x5 according to EN 10210-2:2006. The material is ductile standard steel S235JR with modulus of elasticity $E = 2.10^{11}$ Pa.




1. *FBD*




2. *Determining the support reactions by using the equilibriums*

$$\sum F_x = 0 \Rightarrow \boxed{C_x = 0 \text{ kN}};$$

$$\sum F_y = 0 \Rightarrow A_y + C_y - F = 0 \Rightarrow A_y + C_y = F$$

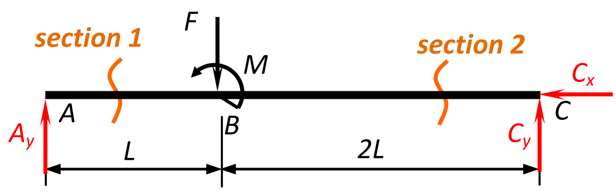
$$\sum M_{z_A} = 0 \Rightarrow M - F.L + C_y.3L = 0 \Rightarrow C_y = \frac{F.L}{3.L} - \frac{M}{3.L} \Rightarrow \boxed{C_y = \frac{F}{3} - \frac{M}{3.L}}$$

From the second equilibrium

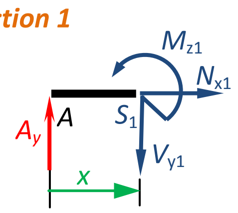
$$\Rightarrow A_y = F - C_y \Rightarrow A_y = F - \left(\frac{F}{3} - \frac{M}{3.L}\right) = \Rightarrow \boxed{A_y = \frac{2F}{3} + \frac{M}{3.L}}$$

3. *Applying the method of the section to determine the internal forces by using the equilibriums*

Two sections have to be made in the regarded example



The left part will be taken for ***section 1***. The internal forces are drawn in the point of the section - $S_1$.

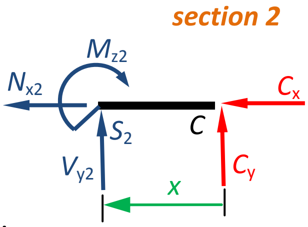



The boundaries of the coordinate $x$ are: $0 \le x \le L$.

Since only $M_z$ will be used in the formula this will be the only internal force to be determined.

$$\sum M_{z_{S1}} = 0 \;\Rightarrow\; M_{z1} - A_y.x = 0 \;\Rightarrow\; M_{z1} = A_y.x \;\Rightarrow\; M_{z1} = \left(\frac{2F}{3} + \frac{M}{3.L}\right)x \;\Rightarrow\; \boxed{M_{z1} = \frac{2F}{3}x + \frac{M}{3.L}x}$$

The right part will be taken for ***section 2***. The internal forces are drawn in the point of the section - $S_2$.




The boundaries of the coordinate $x$ are: $0 \le x \le 2L$.

Again only $M_Z$ will be determined.

$$\sum M_{z_{S2}} = 0 \;\Rightarrow\; M_{z2} - C_y.x = 0 \;\Rightarrow\; M_{z2} = C_y.x \;\Rightarrow\; M_{z2} = \left(\frac{F}{3} - \frac{M}{3.L}\right)x \;\Rightarrow\; \boxed{M_{z2} = \frac{F}{3}x - \frac{M}{3.L}x}.$$

4. *Determining the derivatives of $M_z$*

The formula for $\delta_B^V$ requires the derivative of $M_{z1}$ and $M_{z2}$ with respect to the force concentrated in point $B$ to be determined.

The equation of $M_{z1}$ is

$$M_{z1} = \frac{2F}{3}x + \frac{M}{3.L}x$$

The derivative of $M_{z1}$ with respect to the concentrated in point $B$ force is:

$$\boxed{\frac{\partial M_{z1}}{\partial F_B} = \frac{\partial M_{z1}}{\partial F} = \frac{2}{3}x}$$

*Some comments on how to find derivatives of the internal moment:*

*In this type of problems the derivative of $M_z$ can be found by taking into account with what is the force F multiplied in the expression for $M_z$ , i.e. in the above expression for $M_{z1}$ F is multiplied with $\frac{2}{3}x$ ($M_{z1} = \frac{2F}{3}x + \frac{M}{3.L}x = \left(\frac{2}{3}x\right).F + \frac{M}{3.L}x$ ) and that is why $\frac{\partial M_{z1}}{\partial F} = \frac{2}{3}x$ .*

The equation of $M_{z2}$ is

$$M_{z2} = \frac{F}{3}x - \frac{M}{3.L}x$$

Following the above explanations the derivative of $M_{z2}$ can be found by accounting that *F* is multiplied with $\frac{1}{3}x$ in the expression for $M_{z2}$ which means that

$$\boxed{\frac{\partial M_{z2}}{\partial F_B} = \frac{\partial M_{z2}}{\partial F} = \frac{1}{3}x}$$

5. *Applying the formula for finding the deflection*

Since the structure consists of 2 sections in the formula for $\delta_B^V$ 2 integrals will be used – one for the first section and one for the second one:

$$\delta_B^V = \int_0^L \frac{M_{z1}}{E.I_z}.\frac{\partial M_{z1}}{\partial F_B}dx + \int_0^{2L} \frac{M_{z2}}{E.I_z}.\frac{\partial M_{z2}}{\partial F_B}dx$$

It has to be minded that the boundaries of each integral coincide with the boundaries of each section.

Substituting the expressions for $M_{z1}$ and $M_{z2}$ and their derivatives with respect to *F* will give

$$\delta_B^V = \int_0^L \frac{\left(\frac{2F}{3}x + \frac{M}{3.L}x\right)}{E.I_z}.\left(\frac{2}{3}x\right)dx + \int_0^{2L} \frac{\left(\frac{F}{3}x - \frac{M}{3.L}x\right)}{E.I_z}.\left(\frac{1}{3}x\right)dx$$

Opening the parenthesis will lead to

$$\delta_B^V = \int_0^L \frac{\left(\frac{4F}{9}x^2 + \frac{2M}{9.L}x^2\right)}{E.I_z}dx + \int_0^{2L} \frac{\left(\frac{F}{9}x^2 - \frac{M}{9.L}x^2\right)}{E.I_z}dx$$

This expression can be divided into 4 integrals

$$\delta_B^V = \int_0^L \frac{\left(\frac{4F}{9}x^2\right)}{E.I_z}dx + \int_0^L \frac{\left(\frac{2M}{9.L}x^2\right)}{E.I_z}dx + \int_0^{2L} \frac{\left(\frac{F}{9}x^2\right)}{E.I_z}dx + \int_0^{2L} \frac{\left(-\frac{M}{9.L}x^2\right)}{E.I_z}dx$$

If all the constants are moved in front of the integrals it will follow

$$\delta_B^V = \frac{4F}{9.E.I_z}\int_0^L x^2dx + \frac{2M}{9.L.E.I_z}\int_0^L x^2dx + \frac{F}{9.E.I_z}\int_0^{2L} x^2dx - \frac{M}{9.L.E.I_z}\int_0^{2L} x^2dx$$

These integrals can be solved by using the following formula

$$\int_0^L x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^L = \frac{L^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{L^{n+1}}{n+1}$$

that is

$$\delta_B^V = \frac{4F}{9.E.I_z}.\frac{L^3}{3} + \frac{2M}{9.L.E.I_z}.\frac{L^3}{3} + \frac{F}{9.E.I_z}.\frac{(2L)^3}{3} - \frac{M}{9.L.E.I_z}.\frac{(2L)^3}{3} = \frac{1}{9.E.I_z}\left(4F.\frac{L^3}{3} + \frac{2M}{L}.\frac{L^3}{3} + F.\frac{(2L)^3}{3} - \frac{M}{L}.\frac{(2L)^3}{3}\right)$$

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be in [N], [m], [Pa].

The moment of inertia of the cross-section $I_z$ can be taken from the tables. For the standard shape 160x160x5 according to EN 10210-2:2006 $I_z$ = 1225 cm$^4$ = 1225.10$^{-8}$ m$^4$.

The loads are $F$ = 18 kN = 18.10$^3$ N, $M$ = 6 kN.m = 6.10$^3$ N.m

$$\delta_B^V = \frac{1}{9.2.10^{11}.1225.10^{-8}}\left(4.18.10^3.\frac{3^3}{3} + \frac{2.6.10^3}{3}.\frac{3^3}{3} + 18.10^3.\frac{(2.3)^3}{3} - \frac{6.10^3}{3}.\frac{(2.3)^3}{3}\right)$$

$$\delta_B^V = \frac{1}{22050.10^3}\left(648.10^3 + 36.10^3 + 1296.10^3 - 144.10^3\right) = \frac{1}{22050.10^3}\left(1836.10^3\right)$$

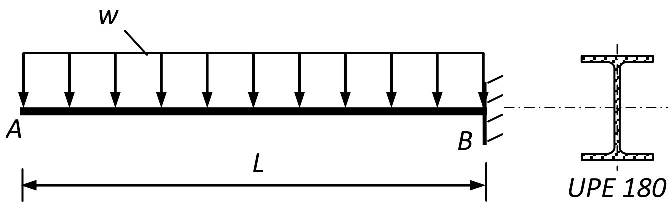
$$\delta_B^V = 0{,}0833\ m$$

The above example shows how to find the vertical deflection $\delta^V$ in a point in which there is a concentrated force.

The next example is related to cases in which the vertical deflection $\delta^V$ has to be determined in a point in which there is no concentrated force.

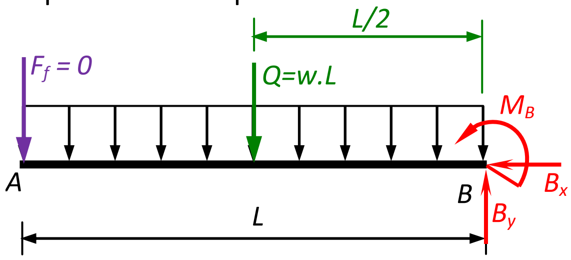
***Example 2***

Find the vertical deflection in point *A* ( $\delta_A^V = ?$ ) if *w* = 20 kN/m, *L* = 2 m. The cross-section is a standard IPE shape – IPE 180. The material is ductile standard steel S235JR with modulus of elasticity E = $2.10^{11}$ Pa.




1. *FBD*

In this example there is no concentrated force in point *A* which makes it impossible to find the derivative of $M_z$. In such cases a fictitious concentrated force $F_f$ which value is equal to zero is added in the point where $\delta^V$ needs to be found. The solution continues according to the steps described in the previous example.




2. *Determining the support reactions by using the equilibriums*

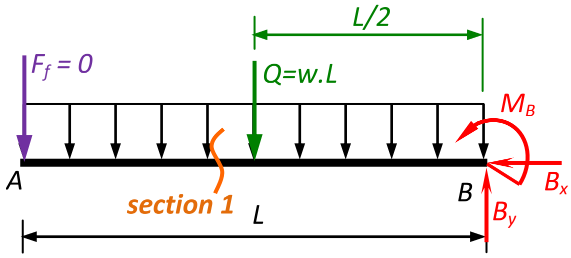
$$\sum F_x = 0 \Rightarrow \boxed{B_x = 0};$$

$$\sum F_y = 0 \Rightarrow B_y - Q - F_f = 0 \Rightarrow \boxed{B_y = w.L + F_f}.$$

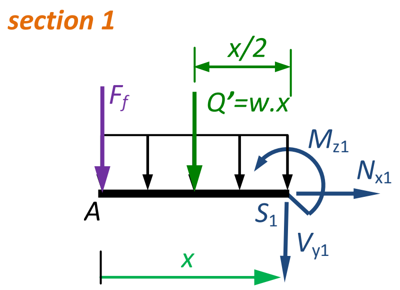
$$\sum M_{z_C} = 0 \Rightarrow M_B + F_f.L + Q.\frac{L}{2} = 0 \Rightarrow \boxed{M_C = -F_f.L - w.L.\frac{L}{2}}$$

3. *Applying the method of the section to determine the internal forces by using the equilibriums*

One section is needed to determine the internal forces.



The left part will be taken for the section. The internal forces are drawn in the point of the section - $S_1$.




The boundaries of the coordinate $x$ are: $0 \le x \le L$.

Since only $M_z$ will be used in the formula this will be the only internal force to be determined.

$$\sum M_{z_{S1}} = 0 \;\Rightarrow\; M_{z1} + F_f.x + Q'.\frac{x}{2} = 0 \;\Rightarrow\; M_{z1} = -F_f.x - w.x.\frac{x}{2} \;\Rightarrow\; \boxed{M_{z1} = -F_f.x - w.\frac{x^2}{2}}$$

4. *Determining the derivatives of $M_z$*

Now the derivative of $M_{z1}$ with respect to the force $F_f$ which is concentrated in point $A$ (the point where $\delta_A^V$ is searched) can be determined. (*To find this it has to be accounted with what $F_f$ is multiplied in the expression of $M_{z1}$*)

$$\boxed{\frac{\partial M_{z1}}{\partial F_A} = \frac{\partial M_{z1}}{\partial F_f} = -x}$$

5. *Applying the formula for finding the deflection*

The formula for $\delta_A^V$ consists of one integral which boundaries are the same as the boundaries of the section:

$$\delta_A^V = \int_0^L \frac{M_{z1}}{E.I_z}.\frac{\partial M_{z1}}{\partial F_A} dx$$

Substituting the expression for $M_{z1}$ and its derivative with respect to $F_f$ will give

$$\delta_A^V = \int_0^L \frac{\left(-F_f.x - w.\frac{x^2}{2}\right)}{E.I_z}.(-x) dx$$

This integral can be simplified when accounting that $F_f$ is zero:

$$\delta_A^V = \int_0^L \frac{\left(-w.\frac{x^2}{2}\right)}{E.I_z}.(-x) dx$$

Opening the parenthesis and moving the constants out of the integral will lead to:

$$\delta_A^V = \frac{w}{2.E.I_z}\int_0^L x^3 dx = \frac{w}{2.E.I_z}\frac{L^4}{4}$$

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be [N], [m], [Pa].

The moment of inertia of the cross-section $I_z$ can be taken from the tables. For the standard IPE shape IPE 180 $I_z$ = 1317 cm$^4$ = 1317.10$^{-8}$ m$^4$.

The load is w = 20 kN/m = 20.10$^3$ N/m.

$$\delta_A^V = \frac{20.10^3}{2.2.10^{11}.1317.10^{-8}}\frac{2^4}{4} \qquad \Rightarrow \qquad \boxed{\delta_A^V = 0{,}0152\ m}$$