How to find vertical deflection in beam structures

Only 2D beam structures will be regarded blow. Only vertical deflection will be discussed.

Vertical deflection is the vertical distance between a point from the undeformed axis of a structure and the same point which lies on the deformed axis. (see the picture below). The formula and the procedure which are described below allow obtaining the vertical deflection in a given point from a structure.

Finding vertical deflection is a task which contains the following steps:

- drawing the Frere Body Diagram (FBD)
- determining the support reactions by using the equilibriums
- applying the method of the section to determine the internal forces by using the equilibriums (only $M_z$ will be determined)
- determining the derivatives of the $M_z$
- applying the formulas for finding deflections

The following formula can be used for finding the vertical deflection in 2D loading in bending:

$$
\delta^y_i = \int \frac{M_z}{E I_z} \frac{\partial M_z}{\partial F_i} \, dx
$$

In the above formula:

- $\delta^y_i$ - vertical deflection in the i-th point,
- $M_z$ – internal moment,
- $E$ – modulus of elasticity (Young’s modulus),
- $I_z$ – moment of inertia of the cross-section,
- $\frac{\partial M_z}{\partial F_i}$ - derivative of $M_z$ with respect to the concentrated force in the i-th point.

The integration is along the length of the beam.

In order to find the **vertical deflection** in a given point there must be a **concentrated force** in this point.

**IT IS IMPORTANT TO MAKE A LETTER SOLUTION (not to replace the parameters with their values) UNTIL SUBSTITUTING IN THE FORMULA!!!**

To demonstrate the above described procedure two examples will be regarded:
**Example 1**

Find the vertical deflection in point $B$ ($\delta_B^V = ?$) if $F = 18$ kN, $M = 6$ kN.m, $L = 3$ m. The cross-section is a standard square hollow section 160x160x5 according to EN 10210-2:2006. The material is ductile standard steel S235JR with modulus of elasticity $E = 2.10^{11}$ Pa.

![Beam diagram]

1. **FBD**

2. **Determining the support reactions by using the equilibriums**

   \[ \sum F_x = 0 \implies C_x = 0 \text{kN} \; ; \]

   \[ \sum F_y = 0 \implies A_y + C_y - F = 0 \implies A_y + C_y = F \]

   \[ \sum M_{z_A} = 0 \implies M - F \cdot L + C_y \cdot 3L = 0 \implies C_y = \frac{F \cdot L}{3L} - \frac{M}{3L} \implies C_y = \frac{F - M}{3L} \]

   From the second equilibrium

   \[ A_y = F - C_y \implies A_y = F - \left( \frac{F}{3} - \frac{M}{3L} \right) = \implies A_y = \frac{2F}{3} + \frac{M}{3L} \]

3. **Applying the method of the section to determine the internal forces by using the equilibriums**

   Two sections have to be made in the regarded example.

   ![Section diagram]
The left part will be taken for section 1. The internal forces are drawn in the point of the section - $S_1$.

The boundaries of the coordinate $x$ are: $0 \leq x \leq L$.

Since only $M_z$ will be used in the formula this will be the only internal force to be determined.

$$\sum M_{z_1} = 0 \Rightarrow M_{z_1} - A_y \cdot x = 0 \Rightarrow M_{z_1} = A_y \cdot x \Rightarrow M_{z_1} = \left(\frac{2F}{3} + \frac{M}{3L}\right) x \Rightarrow M_{z_1} = \frac{2F}{3} x + \frac{M}{3L} x$$

The right part will be taken for section 2. The internal forces are drawn in the point of the section - $S_2$.

The boundaries of the coordinate $x$ are: $0 \leq x \leq 2L$.

Again only $M_z$ will be determined.

$$\sum M_{z_2} = 0 \Rightarrow M_{z_2} - C_y \cdot x = 0 \Rightarrow M_{z_2} = C_y \cdot x \Rightarrow M_{z_2} = \left(\frac{F}{3} - \frac{M}{3L}\right) x \Rightarrow M_{z_2} = \frac{F}{3} x - \frac{M}{3L} x$$

4. Determining the derivatives of $M_z$

The formula for $\delta_{iz}$ requires the derivative of $M_{z1}$ and $M_{z2}$ with respect to the force concentrated in point $B$ to be determined.

The equation of $M_{z1}$ is

$$M_{z1} = \frac{2F}{3} x + \frac{M}{3L} x$$

The derivative of $M_{z1}$ with respect to the concentrated in point $B$ force is:

$$\frac{\partial M_{z1}}{\partial F} = \frac{\partial M_{z1}}{\partial F} = \frac{2}{3} x$$
Some comments on how to find derivatives of the internal moment:

In this type of problems the derivative of $M_z$ can be found by taking into account with what is the force $F$ multiplied in the expression for $M_z$, i.e. in the above expression for $M_{z1}$ $F$ is multiplied with $\frac{2}{3}x$ and that is why $\frac{\partial M_{z1}}{\partial F} = \frac{2}{3}x$.

The equation of $M_{z2}$ is

$$M_{z2} = \frac{F}{3}x - \frac{M}{3L}x$$

Following the above explanations the derivative of $M_{z2}$ can be found by accounting that $F$ is multiplied with $\frac{1}{3}x$ in the expression for $M_{z2}$ which means that

$$\frac{\partial M_{z2}}{\partial F} = \frac{1}{3}x$$

5. Applying the formula for finding the deflection

Since the structure consists of 2 sections in the formula for $\delta_b^\nu$ 2 integrals will be used – one for the first section and one for the second one:

$$\delta_b^\nu = \int_0^1 \frac{M_{z1}}{EJ_z} \frac{\partial M_{z1}}{\partial F} dx + \int_0^2 \frac{M_{z2}}{EJ_z} \frac{\partial M_{z2}}{\partial F} dx$$

It has to be minded that the boundaries of each integral coincide with the boundaries of each section.

Substituting the expressions for $M_{z1}$ and $M_{z2}$ and their derivatives with respect to $F$ will give

$$\delta_b^\nu = \int_0^1 \left(\frac{2F}{3}x + \frac{M}{3L}x\right) \frac{2}{3}x dx + \int_0^2 \left(\frac{F}{3}x - \frac{M}{3L}x\right) \frac{1}{3}x dx$$

Opening the parenthesis will lead to

$$\delta_b^\nu = \int_0^1 \left(\frac{4F}{9}x^3 + \frac{2M}{9L}x^2\right) dx + \int_0^2 \left(\frac{F}{9}x^2 - \frac{M}{9L}x^2\right) dx$$

This expression can be divided into 4 integrals
\[ \delta_y = \int_0^l \left( \frac{4F}{9EJ_z} \right) x^4 \, dx + \int_0^l \left( \frac{2M}{9ELz} \right) x^2 \, dx + \int_0^{2l} \left( \frac{F}{9EJ_z} \right) x^2 \, dx + \int_0^{2l} \left( \frac{M}{9ELz} \right) x^2 \, dx \]

If all the constants are moved in front of the integrals it will follow

\[ \delta_y = \frac{4F}{9EJ_z} \int_0^l x^4 \, dx + \frac{2M}{9ELz} \int_0^l x^2 \, dx + \frac{F}{9EJ_z} \int_0^{2l} x^2 \, dx - \frac{M}{9ELz} \int_0^{2l} x^2 \, dx \]

These integrals can be solved by using the following formula

\[ \int_0^l x^n \, dx = \frac{x^{n+1}}{n+1} \]

that is

\[ \delta_y = \frac{4F}{9EJ_z} \left( \frac{l^5}{3} \right) + \frac{2M}{9ELz} \left( \frac{l^3}{3} \right) + \frac{F}{9EJ_z} \left( \frac{2l^3}{3} \right) - \frac{M}{9ELz} \left( \frac{2l^3}{3} \right) = \frac{1}{9EJ_z} \left( \frac{4F}{3} \cdot \frac{l^3}{3} + \frac{2M}{3} \cdot \frac{l^3}{3} + \frac{F}{3} \cdot \frac{(2l)^3}{3} - \frac{M}{3} \cdot \frac{(2l)^3}{3} \right) \]

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be in [N], [m], [Pa].

The moment of inertia of the cross-section \( I_z \) can be taken from the tables. For the standard shape 160x160x5 according to EN 10210-2:2006 \( I_z = 1225 \text{ cm}^4 = 1225 \times 10^{-8} \text{ m}^4 \).

The loads are \( F = 18 \text{ kN} = 18 \times 10^3 \text{ N} \), \( M = 6 \text{ kN.m} = 6 \times 10^3 \text{ N.m} \)

\[ \delta_y = \frac{1}{9.2 \times 10^{11}.1225.10^{-8}} \left( 4.18 \times 10^3 \cdot \frac{3^3}{3} + 2.6 \times 10^3 \cdot \frac{3^3}{3} + 18.10^3 \cdot \frac{(2.3)^3}{3} - 6.10^4 \cdot \frac{(2.3)^3}{3} \right) \]

\[ \delta_y = \frac{1}{22050.10^3} \left( 648.10^3 + 36.10^3 + 1296.10^3 - 144.10^3 \right) = \frac{1}{22050.10^3} (1836.10^3) \]

\[ \delta_y = 0.0833 \text{ m} \]

The above example shows how to find the vertical deflection \( \delta_y \) in a point in which there is a concentrated force.

The next example is related to cases in which the vertical deflection \( \delta_y \) has to be determined in a point in which there is no concentrated force.
Example 2

Find the vertical deflection in point \( A \) \( (\delta_A = \text{?}) \) if \( w = 20 \text{ kN/m} \), \( L = 2 \text{ m} \). The cross-section is a standard IPE shape – IPE 180. The material is ductile standard steel S235JR with modulus of elasticity \( E = 2.10^{11} \text{ Pa} \).

1. **FBD**

In this example there is no concentrated force in point \( A \) which makes it impossible to find the derivative of \( M_z \). In such cases a fictitious concentrated force \( F_f \) which value is equal to zero is added in the point where \( \delta^y \) needs to be found. The solution continues according to the steps described in the previous example.

2. **Determining the support reactions by using the equilibriums**

\[
\sum F_x = 0 \Rightarrow B_x = 0 ;
\]

\[
\sum F_y = 0 \Rightarrow B_y - Q - F_f = 0 \Rightarrow B_y = wL + F_f .
\]

\[
\sum M_{zC} = 0 \Rightarrow M_b + F_f L + Q \frac{L}{2} = 0 \Rightarrow M_c = -F_f L - wL \frac{L}{2}
\]

3. **Applying the method of the section to determine the internal forces by using the equilibriums**

One section is needed to determine the internal forces.
The left part will be taken for the section. The internal forces are drawn in the point of the section - S₁.

The boundaries of the coordinate \( x \) are: \( 0 \leq x \leq L \).

Since only \( M_z \) will be used in the formula this will be the only internal force to be determined.

\[
\sum M_{z1} = 0 \Rightarrow M_{z1} + F_j \cdot x + Q' \cdot \frac{x}{2} = 0 \Rightarrow M_{z1} = -F_j \cdot x - w \cdot x \cdot \frac{x}{2} \Rightarrow M_{z1} = -F_j \cdot x - w \cdot \frac{x^2}{2}
\]

4. Determining the derivatives of \( M_z \)

Now the derivative of \( M_{z1} \) with respect to the force \( F_f \) which is concentrated in point A (the point where \( \delta'^y_A \) is searched) can be determined. (To find this it has to be accounted with what \( F_f \) is multiplied in the expression of \( M_{z1} \))

\[
\frac{\partial M_{z1}}{\partial F_A} = \frac{\partial M_{z1}}{\partial F_f} = -x
\]

5. Applying the formula for finding the deflection

The formula for \( \delta'^y_A \) consists of one integral which boundaries are the same as the boundaries of the section:

\[
\delta'^y_A = \int_{0}^{L} \frac{M_{z1} \cdot \partial M_{z1}}{EJ_z \cdot \partial F_A} \, dx
\]

Substituting the expression for \( M_{z1} \) and its derivative with respect to \( F_f \) will give

\[
\delta'^y_A = \int_{0}^{L} \left( \frac{-F_j \cdot x - w \cdot \frac{x^2}{2}}{EJ_z} \right) \, (-x) \, dx
\]

This integral can be simplified when accounting that \( F_f \) is zero:

\[
\delta'^y_A = \int_{0}^{L} \left( \frac{-w \cdot \frac{x^2}{2}}{EJ_z} \right) \, (-x) \, dx
\]
Opening the parenthesis and moving the constants out of the integral will lead to:

\[ \delta_\nu = \frac{w}{2.E.I_x} \int_0^L x^3 \, dx = \frac{w L^4}{2.E.I_x} \]

Now the parameters can be replaced with their values from the condition. It has to be minded that the units must be [N], [m], [Pa].

The moment of inertia of the cross-section \( I_x \) can be taken from the tables. For the standard IPE shape IPE 180 \( I_z = 1317 \text{ cm}^4 = 1317.10^{-8} \text{ m}^4 \).

The load is \( w = 20 \text{ kN/m} = 20.10^3 \text{ N/m} \).

\[ \delta_\nu = \frac{20.10^3}{2.2.10^{11}.1317.10^{-8}} \frac{2^4}{4} \quad \Rightarrow \quad \delta_\nu = 0.0152 \text{ m} \]