How to find stresses in beam structures loaded in bending

Only 2D beam structures will be regarded below.

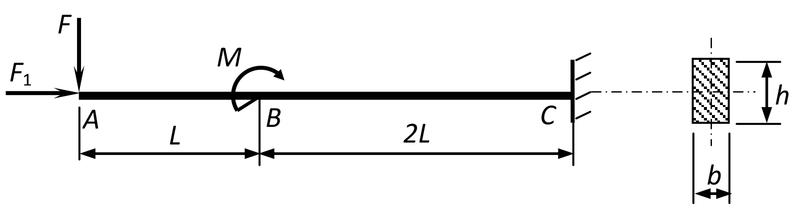
Finding the stress in a structure is a task which consists of several steps:

- drawing the Free Body Diagram (FBD)
- determining the support reactions by using the equilibriums
- applying the method of the section to determine the internal forces by using the equilibriums
- drawing the internal forces diagrams and determining where is the critical section
- obtaining the geometrical properties of the construction cross-section (area, moments of inertia)
- applying the formulas for finding the stresses
- estimation of the construction

To demonstrate the above described procedure the following two examples will be regarded:

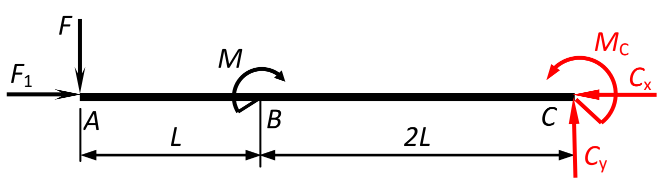
Example 1 – structure with non-standard cross-section

Determine the normal stresses σ_x in the structure and check whether it can bear the loading if F = 10 kN, F_1 = 50 kN, M = 50 kN.m, L = 2 m. The cross-section is a rectangle with dimensions h = 120 mm, b = 50 mm. The material is ductile standard steel S355JR with yielding stress σ_y = 355 MPa. Use a factor of safety FS = 1,29.



1. *FBD*

The support has to be removed and replaced with support reactions. Since this is a 2D structure and the support in point *C* is fixed there will be three support reactions which will be drawn instead of the support.



2. *Determining the support reactions by using the equilibriums*

$$\sum F_x = 0 \;\Rightarrow\; F_1 - C_x = 0 \;\Rightarrow\; C_x = F_1 \;\Rightarrow\; \boxed{C_x = 50\text{ kN}};$$

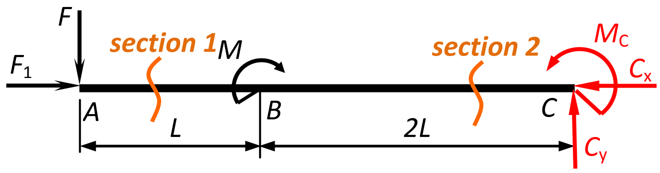
$$\sum F_y = 0 \;\Rightarrow\; C_y - F = 0 \;\Rightarrow\; C_y = F \;\Rightarrow\; \boxed{C_y = 10\text{ kN}}.$$

$$\sum M_{z_C} = 0 \;\Rightarrow\; M_C - M + F.3L = 0 \;\Rightarrow\; M_C = M - F.3L \;\Rightarrow\; M_C = 50 - 10.3.2 \;\Rightarrow\; \boxed{M_C = -10\text{ kN.m}}$$

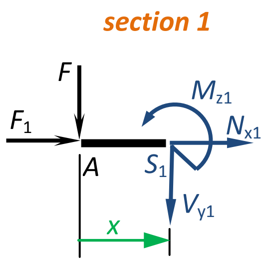
3. *Applying the method of the section to determine the internal forces by using the equilibriums*

Each intermediate concentrated load plays the role of a boundary before and after which a section has to be done. The same role plays the intermediate beginning or ending of a distributed load.

In the regarded example the moment *M* located in point *B* divides the structure into two parts and for each part a section must be done in order to investigate the internal forces.



For ***section 1*** the left part will be taken (that means everything to the right of this section will be removed). The internal forces are drawn in the point of the section - S_1.



A coordinate x is used to determine the position of the section (the exact position of the section is not given). It starts from point *A* and changes to the right. The boundaries of this coordinate are $0 \le x \le L$.

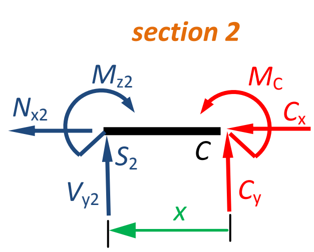
The equilibriums are applied to determine the internal forces.

$$\sum F_x = 0 \;\Rightarrow\; F_1 + N_{x1} = 0 \;\Rightarrow\; N_{x1} = -F_1 \;\Rightarrow\; \boxed{N_{x1} = -50\text{ kN}};$$

$$\sum F_y = 0 \;\Rightarrow\; F + V_{y1} = 0 \;\Rightarrow\; V_{y1} = -F \;\Rightarrow\; \boxed{V_{y1} = -10\text{kN}}.$$

$$\sum M_{z_{S1}} = 0 \;\Rightarrow\; F.x + M_{z1} = 0 \;\Rightarrow\; M_{z1} = -F.x \;\Rightarrow\; \boxed{M_{z1} = -10.x}$$

For ***section 2*** the right part will be taken (everything to the left will be removed). The internal forces are drawn in the point of the section - S_2.



A coordinate x is again used to determine the position of the section but for this section it starts from point *C* and changes to the left and its boundaries are $0 \le x \le 2L$.

The equilibriums are applied to determine the internal forces.

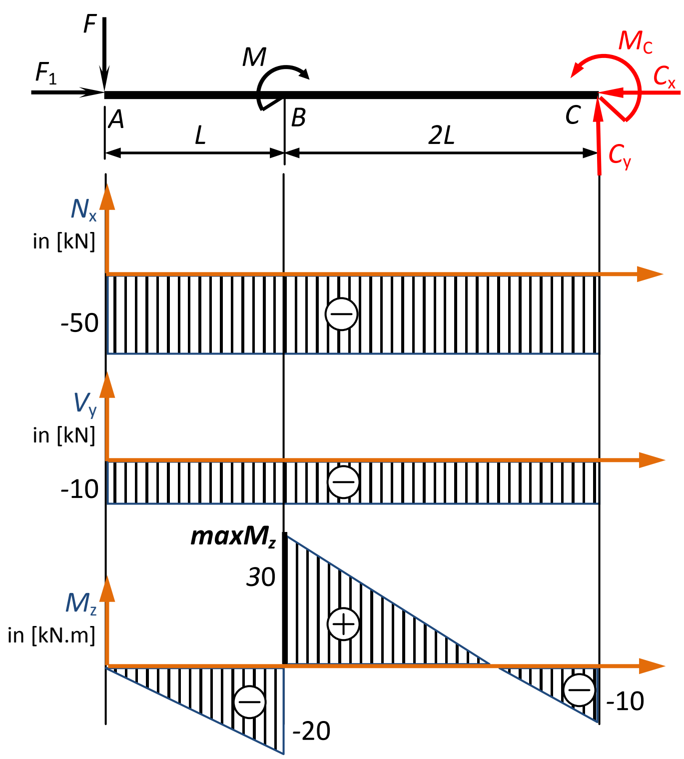
$$\sum F_x = 0 \Rightarrow N_{x2} + C_x = 0 \Rightarrow N_{x2} = -C_x \Rightarrow \boxed{N_{x2} = 50 \text{ kN}};$$

$$\sum F_y = 0 \Rightarrow C_y + V_{y2} = 0 \Rightarrow V_{y2} = -C_y \Rightarrow \boxed{V_{y2} = -10 \text{ kN}};$$

$$\sum M_{z_{S2}} = 0 \Rightarrow M_{z2} - M_C - C_y.x = 0 \Rightarrow M_{z2} = M_C + C_y.x \Rightarrow \boxed{M_{z2} = -10 + 10.x}.$$

4. *Drawing the internal forces diagrams and determining where is the critical section*

Section 1	**Section 2**
$N_{x1} = -50$ kN - constant	$N_{x2} = 50$ kN - constant
$V_{y1} = -10$kN - constant	$V_{y2} = -10$ kN - constant
$M_{z1} = -10.x$ - linear function $0 \le x \le L$ for $x = 0 \Rightarrow M_{z1} = 0$kN.m for $x = L = 2$m $\Rightarrow M_{z1} = -20$kN.m	$M_{z2} = -10 + 10.x$ - linear function $0 \le x \le 2L$ for $x = 0 \Rightarrow M_{z2} = -10$kN.m for $x = 2L = 4$m $\Rightarrow M_{z2} = 30$kN.m



Since only the normal stresses σ_x have to be determined the internal force V_y is neglected.

The critical cross-section in the structure is the point where the internal forces have maximum. Since N_x is constant in the regarded example it follows that the critical cross-section is in point *B* where M_z has maximum – maxM_z = 30 kN.m.

The further calculations of the stresses in the structure will be done only for the critical cross-section.

5. Obtaining the geometrical properties of the construction cross-section

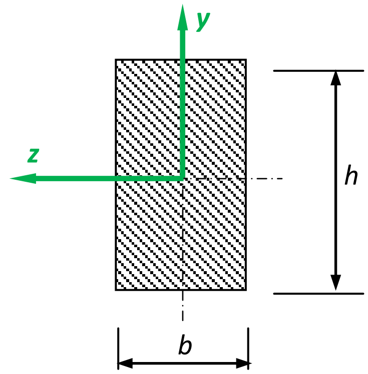
The cross-section of the structure is a rectangle with dimensions h = 120 mm, b = 50 mm.

Using the formulas for the area and moments of inertia of a rectangle (*see Strength of Materials Handbook, page 6*) it can be written

Area - $A = b.h = 120.50 = 6000\text{mm}^2$

Moment of inertia about *y* axis - $I_y = \dfrac{b^3.h}{12} = \dfrac{(50)^3.120}{12} = 1250000\text{mm}^4$

Moment of inertia about *z* axis - $I_z = \dfrac{b.h^3}{12} = \dfrac{50.(120)^3}{12} = 7200000\text{mm}^4$

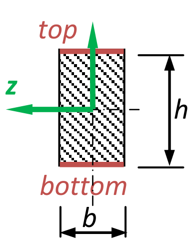


6. Applying the formulas for finding the stresses

The formula for the normal stresses σ_x is

$$\sigma_x = \frac{N_x}{A} - \frac{M_z}{I_z}.y$$

where y is a coordinate in the cross-section which origin is in the centrioid and it changes in vertical direction. The more far away from the centroid the bigger is the coordinate which means that the stresses increase in vertical direction in the cross-section. The maximum stresses will be calculated for the top and the bottom points of the cross-section. These points are called critical points in the critical cross-section.



The coordinate of the top points is $y_t = \dfrac{h}{2} = \dfrac{120}{2} = 60\text{mm}$

The coordinate of the bottom points is $y_b = -\dfrac{h}{2} = -\dfrac{120}{2} = -60\text{mm}$

In order to obtain the maximum values of the stresses the internal forces from the critical cross-section have to be used. In this way the formula for calculation of the maximum stress is

$$\max\sigma_x^t = \frac{N_x}{A} - \frac{\max M_z}{I_z}.y_t \qquad \text{or} \qquad \max\sigma_x^b = \frac{N_x}{A} - \frac{\max M_z}{I_z}.y_b$$

Which of these two formulas produces the highest stresses depends on the signs of the internal forces.

Stress is a unit which dimension is Pa = N/m^2. That is why in the formula for σ_x all units must be in N and m (*see Strength of Materials Handbook, page 39 if you cannot convert units*). That means:

In the critical cross-section

N_x = -50 kN = -50.10^3 N

maxM_z = 30 kN.m = 30.10^3 N.m

Properties of the cross-section

$$A = 6000\text{mm}^2 = 6000.10^{-6}\,\text{m}^2$$

$$I_z = 7200000\text{mm}^4 = 7200000.10^{-12} = 7{,}2.10^{-6}\,\text{m}^4$$

Coordinates of the critical points

$$y_t = 60\text{mm} = 60.10^{-3}\,\text{m}$$

$$y_b = -60\text{mm} = -60.10^{-3}\,\text{m}$$

Now the formula for the normal stresses σ_x ***in the critical points from the critical cross-section*** can be applied

- for the top points

$$\max\sigma_x^t = \frac{N_x}{A} - \frac{\max M_z}{I_z}.y_t = \frac{-50.10^3}{6000.10^{-6}} - \frac{30.10^3}{7{,}2.10^{-6}}.60.10^{-3} = -0{,}0083.10^9 - 250.10^6 =$$
$$= -8{,}3.10^6 - 250.10^6 = -258{,}3.10^6\,\text{Pa} = -258{,}3\text{MPa}$$

- for the bottom points

$$\max\sigma_x^b = \frac{N_x}{A} - \frac{\max M_z}{I_z}.y_b = \frac{-50.10^3}{6000.10^{-6}} - \frac{30.10^3}{7{,}2.10^{-6}}.\left(-60.10^{-3}\right) = -0{,}0083.10^9 + 250.10^6 =$$
$$= -8{,}3.10^6 + 250.10^6 = 241{,}7.10^6\,\text{Pa} = 241{,}7\text{MPa}$$

7. *Estimation of the construction*

The results show that the maximum stresses are in the top points of the critical cross-section

$$\max\sigma_x = \max\sigma_x^t = -258{,}3\text{MPa}$$

The negative value shows that this stress is compressive.

Now the construction can be estimated.

Since the material of the structure is standard steel S355JR with yielding stress σ_y = 355 MPa (*see Strength of Materials Handbook, page 34*) and the factor of safety is FS = 1,29 the allowable stresses σ_{all} can be calculated.

$$\sigma_{all} = \frac{\sigma_y}{FS} = \frac{355}{1,29} = 275,2\text{MPa}$$

The allowable stresses are used as criteria for estimation of the construction.

If the material is ductile

$$\left|\max \sigma_x\right| \le \sigma_{all}$$

If the material is brittle

$$\max \sigma_x^{positive} \le \sigma_{all}^{tension}$$
$$\max \sigma_x^{negative} \le \sigma_{all}^{compression}$$

The S355JR standard steel is a ductile material and

$$\left|\max \sigma_x\right| = 258,3\text{MPa} \le \sigma_{all}$$

which means that ***<u>the structure will resist the applied loads</u>***.

A. Calculation of normal stresses by using the section modulus (only for ductile materials)

Sometimes for fast estimation of structures the following formula is used

$$\max \sigma_x = \frac{\left|N_x\right|}{A} + \frac{\left|\max M_z\right|}{S_z}$$

where S_z is the section modulus which is determined by

$$S_z = \frac{I_z}{\left|y_{\max}\right|}$$

In the regarded example $y_{\max} = y_t = y_b$.

The section modulus of the cross-section is

$$S_z = \frac{I_z}{\left|y_{\max}\right|} = \frac{I_z}{y_t} = \frac{I_z}{\left|y_b\right|} = \frac{7,2.10^{-6}}{60.10^{-3}} = 0,12.10^{-3}\,\text{m}^3$$

Now the maximum stresses will be

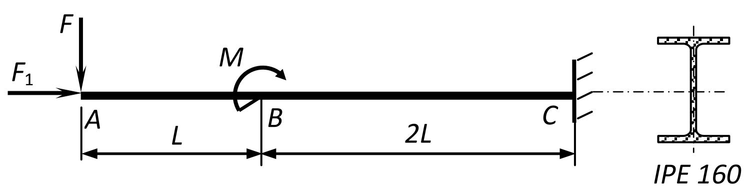
$$\max\sigma_x = \frac{|N_x|}{A} + \frac{|\max M_z|}{S_z} = \frac{50.10^3}{6000.10^{-6}} + \frac{30.10^3}{0,12.10^{-3}} = 0,0083.10^9 + 250.10^6 =$$
$$= 8,3.10^6 + 250.10^6 = 258,5.10^6\,\text{Pa} = 258,3\,\text{MPa}$$

This is the same result as the result for the top point of the critical cross-section. The difference here is that by using this formula it cannot be determined whether the maximum stress is tensile or compressive and in which points (top or bottom) it is.

The next example will regard the same structure with the same loading but the cross-section will be a standard IPE shape (*see Strength of Materials Handbook, page 9 and 10*).

Example 2 – structure with standard cross-section

Determine the normal stresses σ_x in the structure and check whether it can bear the loading if F = 10 kN, F_1 = 50 kN, M = 50 kN.m, L = 2 m. The cross-section is a standard IPE shape – IPE 160. The material is ductile standard steel S355JR with yielding stress σ_y = 355 MPa. Use a factor of safety FS = 1,29.



Steps 1 to 4 for this example are the same as for *Example 1* which means that the critical cross-section is in point *B* where M_z has maximum – maxM_z = 30 kN.m.

5. *Obtaining the geometrical properties of the construction cross-section*

The cross-section properties of standard shapes are given in tables (*see Strength of Materials Handbook, page 9 and 10*).

For IPE 160 they are the following:

Area - $A = 20,1\text{cm}^2$

Moment of inertia about *y* axis - $I_y = 68,3\text{cm}^4$

Moment of inertia about *z* axis - $I_z = 869\text{cm}^4$

Section modulus about *y* axis - $S_y = 16,7\text{cm}^3$

Section modulus about *z* axis - $S_z = 109\text{cm}^3$

6. *Applying the formulas for finding the stresses*

For this example the formula for the normal stresses which uses the section modulus will be applied since the material of the structure is ductile.

$$\max\sigma_x = \frac{|N_x|}{A} + \frac{|\max M_z|}{S_z}$$

In order to apply the formula all units have to be converted in N and m:

In the critical cross-section

N_x = -50 kN = -50.10^3 N ; maxM_z = 30 kN.m = 30.10^3 N.m

Properties of the cross-section

$$A = 20{,}1\text{cm}^2 = 20{,}1.10^{-4}\,\text{m}^2$$

$$S_z = 109\,\text{cm}^3 = 109.10^{-6}\,\text{m}^3$$

Now the stresses can be calculated:

$$\max\sigma_x = \frac{|N_x|}{A} + \frac{|\max M_z|}{S_z} = \frac{50.10^3}{20{,}1.10^{-4}} + \frac{30.10^3}{109.10^{-6}} = 2{,}49.10^7 + 0{,}2752.10^9 =$$
$$= 24{,}9.10^6 + 275{,}2.10^6 = 300{,}1.10^6\,\text{Pa} = 300{,}1\text{MPa}$$

7. *Estimation of the construction*

The allowable stresses σ_{all} are

$$\sigma_{all} = \frac{\sigma_y}{FS} = \frac{355}{1{,}29} = 275{,}2\text{MPa}$$

Since for this structure

$$|\max\sigma_x| = 300{,}1\text{MPa} \geq \sigma_{all}$$

it can be concluded that ***the structure is not appropriate for the applied loads and bigger IPE shape has to be used.***